

ON THE SUBINVARIANCE OF UNIFORM DOMAINS IN METRIC SPACES

YAXIANG LI, MANZI HUANG, XIANTAO WANG*, AND QINGSHAN ZHOU

ABSTRACT. Suppose that X and Y are quasiconvex and complete metric spaces, that $G \subset X$ and $G' \subset Y$ are domains, and that $f : G \rightarrow G'$ is a homeomorphism. Our main result is the following subinvariance property of the class of uniform domains: Suppose both f and f^{-1} are weakly quasisymmetric mappings and G' is a quasiconvex domain. Then the image $f(D)$ of every uniform subdomain D in G under f is uniform. The subinvariance of uniform domains with respect to freely quasiconformal mappings or quasihyperbolic mappings is also studied with the additional condition that both G and G' are locally John domains.

1. INTRODUCTION AND MAIN RESULTS

The quasihyperbolic metric (briefly, QH metric) was introduced by Gehring and his students Palka and Osgood in the 1970's [12, 13] in the setting of Euclidean spaces \mathbb{R}^n ($n \geq 2$). Since its first appearance, the quasihyperbolic metric has become an important tool in the geometric function theory of Euclidean spaces, especially, in the study of quasiconformal and quasisymmetric mappings. From late 1980's onwards, Väisälä has developed the theory of (dimension) free quasiconformal mappings (briefly, free theory) in Banach spaces, which is based on properties of the quasihyperbolic metric [36, 37, 38, 39, 41]. The main advantage of this approach over generalizations based on the conformal modulus (see [16] and references therein) is that it does not make use of volume integrals, thus allowing the study of quasiconformality in infinite dimensional Banach spaces.

The class of quasisymmetric mappings on the real axis was first introduced by Beurling and Ahlfors [4], who found a way to obtain a quasiconformal extension of a quasisymmetric self-mapping of the real axis to a self-mapping of the upper half-plane. This idea was later generalized by Tukia and Väisälä, who studied quasisymmetric mappings between metric spaces [32]. In 1998, Heinonen and Koskela [18] proved a remarkable result, showing that the concepts of quasiconformality and quasisymmetry are quantitatively equivalent in a large class of metric spaces, which includes Euclidean spaces. Also, Väisälä proved the quantitative equivalence between free quasiconformality and quasisymmetry of homeomorphisms between two Banach spaces. See [41, Theorem 7.15]. Against this background, it is not surprising

File: Subinvariance-with-pics-revised15-1-12.tex, printed: 2015-2-11, 4.22

2000 *Mathematics Subject Classification*. Primary: 30C65, 30F45; Secondary: 30C20.

Key words and phrases. Subinvariance, uniform domain, weak quasisymmetry, free quasiconformality, quasihyperbolic homomorphism, locally John domain.

* Corresponding author.

The research was partly supported by NNSF of China.

that the study of quasisymmetry in metric spaces has recently attracted significant attention.

The main goal of this paper is to establish the subinvariance of uniform domains in suitable metric spaces with respect to quasisymmetric mappings, freely quasiconformal mappings and quasihyperbolic mappings. We start by recalling some basic definitions. Through this paper, we always assume that X and Y are metric spaces. We follow the notations and terminology of [17, 18, 25, 33, 41].

Definition 1. A homeomorphism f from X to Y is said to be

- (1) *quasiconformal* if there is a constant $H < \infty$ such that

$$(1.1) \quad \limsup_{r \rightarrow 0} \frac{L_f(x, r)}{l_f(x, r)} \leq H$$

for all $x \in X$;

- (2) *quasisymmetric* if there is a constant $H < \infty$ such that

$$(1.2) \quad \frac{L_f(x, r)}{l_f(x, r)} \leq H$$

for all $x \in X$ and all $r > 0$,

where $L_f(x, r) = \sup_{|y-x| \leq r} \{|f(y) - f(x)|\}$ and $l_f(x, r) = \inf_{|y-x| \geq r} \{|f(y) - f(x)|\}$.

Here and in what follows, we always use $|x - y|$ to denote the distance between x and y .

Definition 2. A homeomorphism f from X to Y is said to be

- (1) η -*quasisymmetric* if there is a homeomorphism $\eta : [0, \infty) \rightarrow [0, \infty)$ such that

$$|x - a| \leq t|x - b| \text{ implies } |f(x) - f(a)| \leq \eta(t)|f(x) - f(b)|$$

for each $t \geq 0$ and for each triple x, a, b of points in X ;

- (2) *weakly H -quasisymmetric* if

$$|x - a| \leq |x - b| \text{ implies } |f(x) - f(a)| \leq H|f(x) - f(b)|$$

for each triple x, a, b of points in X .

Remark 1. The following observations follow immediately from Definitions 1 and 2.

- (1) The quasisymmetry implies the quasiconformality;
- (2) A homeomorphism f from X to Y is quasisymmetric with coefficient H defined by Definition 1(2) if and only if it is weakly H -quasisymmetric;
- (3) The η -quasisymmetry implies the weak H -quasisymmetry with $H = \eta(1)$. Obviously, $\eta(1) \geq 1$. In general, the converse is not true (cf. [41, Theorem 8.5]). See also [22] for the related discussions.

The definition of free quasiconformality is as follows.

Definition 3. Let $G \subsetneq X$ and $G' \subsetneq Y$ be two domains (open and connected), and let $\varphi : [0, \infty) \rightarrow [0, \infty)$ be a homeomorphism with $\varphi(t) \geq t$. We say that a homeomorphism $f : G \rightarrow G'$ is

(1) φ -semisolid if

$$k_{G'}(f(x), f(y)) \leq \varphi(k_G(x, y))$$

for all $x, y \in G$;

(2) φ -solid if both f and f^{-1} are φ -semisolid;

(3) *freely φ -quasiconformal* (φ -FQC in brief) or *fully φ -solid* if f is φ -solid in every subdomain of G ,

where $k_G(x, y)$ denotes the quasihyperbolic distance of x and y in G . See Section 2 for the precise definitions of $k_G(x, y)$ and other notations and concepts in the rest of this section.

It follows from [9, Remark, p. 121] and [34, Theorem 5.6] that uniform domains are subinvariant with respect to quasiconformal mappings in \mathbb{R}^n ($n \geq 2$). By this, we mean that if $f : G \rightarrow G'$ is a K -quasiconformal mapping, where G and G' are domains in \mathbb{R}^n , and if G' is c -uniform, then $D' = f(D)$ is c' -uniform for every c -uniform subdomain $D \subset G$, where $c' = c'(c, K, n)$ which means that the constant c' depends only on the coefficient c of the uniformity of D , the coefficient K of quasiconformality of f and the dimension n of the Euclidean space \mathbb{R}^n . In the free theory, Väisälä also studied this property of uniform domains in Banach spaces and proved the following result.

Theorem A. ([40, Theorem 2.44]) *Suppose that $G \subset E$ and $G' \subset E'$, where E and E' are Banach spaces with dimension at least 2, that the domains G' and $D \subset G$ are c -uniform, and that $f : G \rightarrow G'$ is M -quasihyperbolic. Then $f(D)$ is c' -uniform with $c' = c'(c, M)$.*

In 2012, Huang, Vuorinen and Wang proved the subinvariance property of uniform domains is also true with respect to freely quasiconformal mappings ([24, Theorem 1]). See [7, 11, 34, 41, 43] for similar discussions in this line.

Our work is motivated by the above ideas which we will extend to the context of weakly quasisymmetric mappings, freely quasiconformal mappings and quasihyperbolic mappings in metric spaces. Our first result is as follows.

Theorem 1. *Suppose that X and Y are quasiconvex and complete metric spaces, that $G \subsetneq X$ is a domain, $G' \subsetneq Y$ is a quasiconvex domain, and that both $f : G \rightarrow G'$ and $f^{-1} : G' \rightarrow G$ are weakly quasisymmetric mappings. For each subdomain D of G , if D is uniform, then $D' = f(D)$ is uniform, where the coefficient of uniformity of D' depends only on the given data of X, Y, G, G', D, f and f^{-1} .*

Here and in what follows, the phrase “the given data of X, Y, G, G', D, f and f^{-1} ” means the data which depends on the given constants which are the coefficients of quasiconvexity of X, Y and G' , the coefficient of uniformity of G and the coefficients of weak quasisymmetry of f and f^{-1} .

Remark 2. *It is worth to mention that in Theorem 1, the domain G' is not required to be “uniform”, and only to be “quasiconvex” (From the definitions in Section 2, we easily see that uniformity implies quasiconvexity). Moreover, we see from the example constructed in the paragraph next to Theorem 1.1 in [43] that the assumption “quasiconvexity” in Theorem 1 is necessary.*

As we have indicated in the second paragraph that quasisymmetry and quasiconformality are quantitatively equivalent for homeomorphisms between \mathbb{R}^n . Also, it follows from [12] and [32] that quasiconformality and free quasiconformality are quantitatively equivalent for homeomorphisms between domains in \mathbb{R}^n . These facts together with [24, Theorem 1] prompt us to conjecture that whether Theorem 1 still holds if we replace the assumption “weakly quasisymmetric mappings” (resp. “ G' being quasiconvex”) by the one “freely quasiconformal mappings” (resp. “ G' being uniform”). Under the extra assumption “ G and G' being locally John”, we have the following partial answer to this problem.

Theorem 2. *Suppose that X and Y are quasiconvex and complete metric spaces, that $G \subsetneq X$ is a non-point-cut and locally John domain, and that $G' \subsetneq Y$ is an uniform and locally John domain. If $f : G \rightarrow G'$ is a freely quasiconformal mapping, then for each uniform subdomain D in G , its image $D' = f(D)$ must be uniform, where the coefficient of uniformity of D' depends only on the given data of X, Y, G, G', D and f .*

Here G is said to be *non-point-cut* if for any $x \in G$, the set $G \setminus \{x\}$ is a subdomain of G .

Remark 3. *When $X = Y = \mathbb{R}^n$, Theorem 2 is the subinvariance of uniform domains with respect to quasiconformal mappings in \mathbb{R}^n ([9, Remark, P.121] and [34, Theorem 5.6]). Also, when X and Y are Banach spaces with dimension at least 2, Theorem 2 coincides with [24, Theorem 1].*

Also, we get the following subinvariance of uniform domains with respect to quasi-hyperbolic mappings.

Theorem 3. *Suppose that X and Y are quasiconvex and complete metric spaces, that $G \subsetneq X$ is a non-point-cut and locally John domain, and that $G' \subsetneq Y$ is an uniform and locally John domain. If $f : G \rightarrow G'$ is a quasihyperbolic mapping, then for each uniform subdomain D in G , its image $D' = f(D)$ must be uniform, where the coefficient of uniformity of D' depends only on the given data of X, Y, G, G', D and f .*

Remark 4. *When X and Y are Banach spaces with dimension at least 2, Theorem 3 coincides with Theorem A, i.e., Theorem 2.44 in [40].*

We also conjecture that whether there is the subinvariance of John domains in suitable metric spaces with respect to weakly quasisymmetric mappings, freely quasiconformal mappings etc. See [15, 21] etc for the related discussions in \mathbb{R}^n .

The rest of this paper is organized as follows. In Section 2, we recall some definitions and preliminary results. In Section 3, the proof of Theorem 1 is given. Section 4 is devoted to the proof of Theorem 2, and the proof of Theorem 3 is presented in Section 5.

2. PRELIMINARIES

In this section, we give the necessary definitions and auxiliary results, which will be used in the proofs of our main results.

Throughout this paper, balls and spheres in metric spaces X are written as

$$\mathbb{B}(a, r) = \{x \in X : |x - a| < r\}, \quad \mathbb{S}(a, r) = \{x \in X : |x - a| = r\}$$

and

$$\overline{\mathbb{B}}(a, r) = \mathbb{B}(a, r) \cup \mathbb{S}(a, r) = \{x \in X : |x - a| \leq r\}.$$

For convenience, given domains $G \subset X$, $G' \subset Y$, a map $f : G \rightarrow G'$ and points x, y, z, \dots in G , we always denote by x', y', z', \dots the images in G' of x, y, z, \dots under f , respectively. Also, we assume that γ denotes an arc in G and γ' the image in G' of γ under f .

2.1. Quasihyperbolic metric, solid arcs and short arcs. In this subsection, we start with the definition of quasihyperbolic metric. If X is a connected metric space and $G \subsetneq X$ is a non-empty open set, then it follows from [25, Remark 2.2] that the boundary of G satisfies $\partial G \neq \emptyset$. Suppose $\gamma \subset G$ denotes a rectifiable arc or a path, its *quasihyperbolic length* is the number:

$$\ell_{k_G}(\gamma) = \int_{\gamma} \frac{|dz|}{\delta_G(z)},$$

where $\delta_G(z)$ denotes the distance from z to ∂G .

For each pair of points x, y in G , the *quasihyperbolic distance* $k_G(x, y)$ between x and y is defined in the following way:

$$k_G(x, y) = \inf \ell_{k_G}(\gamma),$$

where the infimum is taken over all rectifiable arcs γ joining x to y in G .

Suppose X is quasiconvex and $G \subsetneq X$. If γ is a rectifiable curve in G connecting x and y , then (see, e.g., the proof of Theorem 2.7 in [25])

$$\ell_{k_G}(\gamma) \geq \log \left(1 + \frac{\ell(\gamma)}{\min\{\delta_G(x), \delta_G(y)\}} \right)$$

and thus,

$$k_G(x, y) \geq \log \left(1 + \frac{|x - y|}{\min\{\delta_G(x), \delta_G(y)\}} \right).$$

Gehring and Palka [13] introduced the quasihyperbolic metric of a domain in \mathbb{R}^n . For the basic properties of this metric we refer to [12]. Recall that a curve γ from x to y is a *quasihyperbolic geodesic* if $\ell_{k_G}(\gamma) = k_G(x, y)$. Each subcurve of a quasihyperbolic geodesic is obviously a quasihyperbolic geodesic. It is known that a quasihyperbolic geodesic between any two points in a Banach space X exists if the dimension of X is finite, see [12, Lemma 1]. This is not true in arbitrary metric spaces (cf. [36, Example 2.9]).

Let us recall a result which is useful for the discussions later on.

Lemma B. ([23, Lemma 2.4]) *Let X be a c -quasiconvex metric space and let $G \subsetneq X$ be a domain. Suppose that $x, y \in G$ and either $|x - y| \leq \frac{1}{3c}\delta_G(x)$ or $k_G(x, y) \leq 1$. Then*

$$(2.1) \quad \frac{1}{2} \frac{|x - y|}{\delta_G(x)} < k_G(x, y) < 3c \frac{|x - y|}{\delta_G(x)}.$$

Here, we say that X is c -*quasiconvex* ($c \geq 1$) if each pair of points $x, y \in X$ can be joined by an arc γ with length $\ell(\gamma) \leq c|x - y|$.

Definition 4. Suppose γ is an arc in a domain $G \subsetneq X$ and X is a rectifiably connected metric space. The arc may be closed, open or half open. Let $\bar{x} = (x_0, \dots, x_n)$, $n \geq 1$, be a finite sequence of successive points of γ . For $h \geq 0$, we say that \bar{x} is h -*coarse* if $k_G(x_{j-1}, x_j) \geq h$ for all $1 \leq j \leq n$. Let $\Phi_k(\gamma, h)$ denote the family of all h -coarse sequences of γ . Set

$$s_k(\bar{x}) = \sum_{j=1}^n k_G(x_{j-1}, x_j)$$

and

$$\ell_{k_G}(\gamma, h) = \sup\{s_k(\bar{x}) : \bar{x} \in \Phi_k(\gamma, h)\}$$

with the agreement that $\ell_{k_G}(\gamma, h) = 0$ if $\Phi_k(\gamma, h) = \emptyset$. Then the number $\ell_{k_G}(\gamma, h)$ is the h -*coarse quasihyperbolic length* of γ .

If X is c -quasiconvex, then $\ell_{k_G}(\gamma, 0) = \ell_{k_G}(\gamma)$ (see, e.g., [5, Proposition A.7 and Remark A.13] and [23, Lemma 2.5]).

Definition 5. Let G be a proper domain in a rectifiably connected metric space X . An arc $\gamma \subset G$ is (ν, h) -*solid* with $\nu \geq 1$ and $h \geq 0$ if

$$\ell_{k_G}(\gamma[x, y], h) \leq \nu k_G(x, y)$$

for all $x, y \in \gamma$.

An arc $\gamma \subset G$ with endpoints x and y is said to be ε -short ($\varepsilon \geq 0$) if

$$\ell_{k_G}(\gamma) \leq k_G(x, y) + \varepsilon.$$

Obviously, by the definition of k_G , we know that for every $\varepsilon > 0$, there exists an arc $\gamma \subset G$ such that γ is ε -short, and it is easy to see that every subarc of an ε -short arc is also ε -short.

Remark 5. For any pair of points x and y in a proper domain G of Banach space E , if the dimension of E is finite, then there exists a quasihyperbolic geodesic in G connecting x and y (see [12, Lemma 1]). But if the dimension of E is infinite, this property is no longer valid (see, e.g., [36, Example 2.9]). In order to overcome this shortcoming in Banach spaces, Väisälä proved the existence of neargeodesics or quasigeodesics (see [37]), and every quasihyperbolic geodesic is a quasigeodesic. See also [31]. In metric spaces, we do not know if this existence property is true or not. However, this existence property plays a very important role in the related discussions. In order to overcome this disadvantage, in this paper, we will exploit the substitution of “quasigeodesics” replaced by “short arcs”. The class of short arcs has been introduced when Väisälä studied properties of Gromov hyperbolic spaces [42] (see also [6, 19]), and as we see that the existence of such class of arcs is obvious in metric spaces. Although, there is no implication between the class of quasigeodesics and the one of short arcs, we will prove that, under certain geometric assumptions, every short arc is a double cone arc.

By a slight modification of the method used in the proof of [37, Lemma 6.21], we get the following result.

Lemma 1. *Suppose that X is a c -quasiconvex metric space and that $G \subsetneq X$ is a domain, and that γ is a (ν, h) -solid arc in G with endpoints x, y such that $\min\{\delta_G(x), \delta_G(y)\} = r \geq 3c|x - y|$. Then there is a constant $\mu_1 = \mu_1(c, \nu)$ such that*

$$\text{diam}(\gamma) \leq \max\{\mu_1|x - y|, 2r(e^h - 1)\},$$

where “diam” means “diameter”.

Proof. Without loss of generality, we assume that $\delta_G(y) \geq \delta_G(x) = r$. Denoting $t = |x - y|$ and applying Lemma B, we get

$$k_G(x, y) \leq 3ct/r.$$

Let $u \in \gamma$. To prove this lemma, it suffices to show that there exists a constant $\mu_1 = \mu_1(c, \nu)$ such that

$$(2.2) \quad |u - x| \leq \max\left\{\frac{\mu_1}{2}|x - y|, r(e^h - 1)\right\}.$$

To this end, we consider two cases. The first case is: $k_G(u, x) \leq h$. Under this assumption, it is easy to see that

$$(2.3) \quad |u - x| \leq (e^{k_G(u, x)} - 1)\delta_G(x) \leq r(e^h - 1).$$

For the remaining case: $k_G(u, x) > h$, we choose a sequence of successive points of γ : $x = x_0, \dots, x_n = u$ such that

$$k_G(x_{j-1}, x_j) = h \quad \text{for } j \in \{1, \dots, n-1\}$$

and

$$0 < k_G(x_{n-1}, x_n) \leq h.$$

Then $n \geq 2$ and

$$(n-1)h \leq \sum_{i=1}^{n-1} k_G(x_{i-1}, x_i) \leq \ell_{k_G}(\gamma, h) \leq \nu k_G(x, y) \leq 3c\nu t/r,$$

which shows that

$$k_G(x, u) \leq \sum_{i=1}^n k_G(x_{i-1}, x_i) \leq nh \leq 6c\nu t/r.$$

Let $s = \frac{t}{r}$. Then $s \leq \frac{1}{3c}$ and

$$\frac{|u - x|}{t} \leq \frac{e^{6c\nu s} - 1}{s}.$$

Obviously, the function $g(s) = \frac{1}{s}(e^{6c\nu s} - 1)$ is increasing in $(0, \frac{1}{3c}]$ and $\lim_{s \rightarrow 0} \frac{e^{6c\nu s} - 1}{s} = 6c\nu$. Letting

$$\mu_1 = 6c(e^{2\nu} - 1)$$

gives

$$(2.4) \quad |u - x| \leq \frac{1}{2}\mu_1 t.$$

It follows from (2.3) and (2.4) that (2.2) holds, and hence the proof of the lemma is complete. \square

Lemma 2. *Suppose that X is a c -quasiconvex metric space and $G \subsetneq X$ is a domain. Suppose, further, that for $x, y \in G$,*

- (1) γ is an ε -short arc in G connecting x and y with $0 < \varepsilon \leq \frac{1}{2}k_G(x, y)$, and
- (2) $|x - y| \leq \frac{1}{3c} \min\{\delta_G(x), \delta_G(y)\}$.

Then

$$\ell(\gamma) \leq \frac{9}{2}ce^{\frac{3}{2}}|x - y|.$$

Proof. Without loss of generality, we assume that $\min\{\delta_G(x), \delta_G(y)\} = \delta_G(x)$. It follows from Lemma B that

$$\log \left(1 + \frac{\ell(\gamma)}{\delta_G(x)} \right) \leq \ell_{k_G}(\gamma) \leq k_G(x, y) + \varepsilon \leq \frac{3}{2}k_G(x, y) \leq \frac{9c}{2} \frac{|x - y|}{\delta_G(x)} \leq \frac{3}{2}.$$

Hence,

$$\frac{\ell(\gamma)}{\delta_G(x)} \leq e^{\frac{3}{2}} - 1,$$

which leads to

$$\frac{\ell(\gamma)}{\delta_G(x)} \leq e^{\frac{3}{2}} \log \left(1 + \frac{\ell(\gamma)}{\delta_G(x)} \right) \leq \frac{9}{2}ce^{\frac{3}{2}} \frac{|x - y|}{\delta_G(x)}.$$

Therefore,

$$\ell(\gamma) \leq \frac{9}{2}ce^{\frac{3}{2}}|x - y|,$$

as required. \square

2.2. Uniform domains, John domains and locally John domains. In 1961, John [26] introduced the twisted interior cone condition with his work on elasticity, and these domains were first called John domains by Martio and Sarvas in [30]. In the same paper, Martio and Sarvas also introduced another class of domains which are the uniform domains. The main motivation for studying these domains was in showing global injectivity properties for locally injective mappings. Since then, many other characterizations of uniform and John domains have been established, see [10, 12, 29, 37, 40, 41], and the importance of these classes of domains in the function theory is well documented (see e.g. [10, 34]). Moreover, John and uniform domains in \mathbb{R}^n enjoy numerous geometric and function theoretic properties that are useful in other many fields of modern mathematical analysis as well (see e.g. [1, 3, 12, 14, 20, 27, 28, 34], and references therein).

We recall the definitions of uniform domains and John domains following closely the notation and terminology of [32, 34, 35, 36, 37] and [29].

Definition 6. A domain G in X is called b -uniform provided there exists a constant b with the property that each pair of points x, y in G can be joined by a rectifiable arc γ in G satisfying

- (1) $\min\{\ell(\gamma[x, z]), \ell(\gamma[z, y])\} \leq b\delta_G(z)$ for all $z \in \gamma$, and

$$(2) \ell(\gamma) \leq b|x - y|,$$

where $\ell(\gamma)$ denotes the length of γ , $\gamma[x, z]$ the part of γ between x and z . At this time, γ is said to be a *double b-cone arc*. Condition (1) (resp. (2)) is called the *cigar condition* (resp. *turning condition*).

If the condition (1) is satisfied, not necessarily (2), then G is said to be a *b-John domain*. At this time, the arc γ is called a *b-cone arc*.

Definition 7. A domain G in X is said to be a *locally a-John domain* if there exists a constant a such that all metric balls $\mathbb{B}(x, r)$ are a -John domains, where $x \in G$ and $0 < r \leq \delta_G(x)$.

We note that all domains which satisfies the strong geodesic condition (see [8] for the definition) in the abstract setting of homogeneous spaces are locally John domains [8, Corollary 3.2]. In particular, all domains in Carnot-Carathéodory metric spaces [8] and Banach spaces [37] are locally John domains.

We remark that for $x \in G$, it is possible that the metric ball $\mathbb{B}(x, r)$ is not contained in G for $0 < r \leq \delta_G(x)$. But, in [23], the authors proved the following.

Lemma C. *Suppose X is a c -quasiconvex metric space and $G \subsetneq X$ is a domain. Then for any rectifiably connected open set $D \subset \mathbb{B}(x, r)$ with $x \in D \cap G$, if $r \leq \delta_G(x)$, then $D \subset G$.*

It follows from Lemma C that the following result is obvious.

Lemma 3. *Suppose X is a c -quasiconvex metric space and $G \subsetneq X$ is a locally John domain. For any $x \in G$, if $0 < r \leq \delta_G(x)$, then $\mathbb{B}(x, r) \subset G$.*

Further, for locally John domains, we have the following estimate for the quasi-hyperbolic metric.

Lemma 4. *Suppose $G \subsetneq X$ is a locally a -John domain. Then for $x, y \in G$ with $|x - y| = t\delta_G(x)$, where $0 < t < 1$, we have*

$$k_G(x, y) \leq 2a \frac{3+t}{1-t}.$$

Proof. Let $x, y \in G$ with $|x - y| = t\delta_G(x)$. Since G is locally a -John, we know from Definition 7 and Lemma 4 that $\mathbb{B}(x, \frac{1+t}{2}\delta_G(x)) \subset G$ is an a -John domain. Then there is a curve $\gamma \in \mathbb{B}(x, \frac{1+t}{2}\delta_G(x))$ such that for any $w \in \gamma$,

$$\min\{\ell(\gamma[y, w]), \ell(\gamma[x, w])\} \leq a\delta_G(w)$$

and

$$\delta_G(w) \geq \delta_G(x) - |x - w| \geq \frac{1}{2}(1-t)\delta_G(x).$$

Let $x_0 \in \gamma$ be the point bisecting the arc length of γ . Then

$$\ell(\gamma) \leq 2a\delta_G(x_0) \leq 2a(\delta_G(x) + |x - x_0|) \leq a(3+t)\delta_G(x),$$

which shows that

$$k_G(x, y) \leq \int_{\gamma} \frac{|dw|}{\delta_G(w)} \leq \frac{2\ell(\gamma)}{(1-t)\delta_G(x)} \leq 2a \frac{3+t}{1-t},$$

as required. \square

Let us recall the following useful property of uniform domains.

Lemma D. ([5, Lemma 3.12]) *Suppose $G \subsetneq X$ is a b -uniform domain in a rectifiable connected metric space X . Then for any $x, y \in G$, we have*

$$k_G(x, y) \leq 4b^2 \log \left(1 + \frac{|x - y|}{\min\{\delta_G(x), \delta_G(y)\}} \right).$$

The following are the analogues of Lemmas 6.10 and 6.11 in [37] in the setting of metric spaces. The proofs are similar.

Lemma E. *Suppose that $G \subsetneq X$ is a b -uniform domain in a rectifiable connected metric space X , and that γ is an arc in $\{x \in G : \delta_G(x) \leq r\}$. If γ is (ν, h) -solid, then*

$$\text{diam}(\gamma) \leq M_1 r,$$

where $M_1 = M_1(b, \nu, h)$.

Lemma F. *For all $b \geq 1$, $\nu \geq 1$ and $h \geq 0$, there are constants $0 < q_0 = q_0(b, \nu, h) < 1$ and $M_2 = M_2(b, \nu, h) \geq 1$ with the following property: Suppose that G is a b -uniform domain and γ is a (ν, h) -solid arc starting at $x_0 \in G$. If γ contains a point u with $\delta_G(u) \leq q_0 \delta_G(x_0)$, then*

$$\text{diam}(\gamma_u) \leq M_2 \delta_G(u),$$

where $\gamma_u = \gamma \setminus \gamma[x_0, u]$.

Now, we are ready to prove an analogue of Lemma 1 for uniform domains.

Lemma 5. *Suppose that X is a c -quasiconvex metric space and that $G \subsetneq X$ is a b -uniform domain, and that γ is a (ν, h) -solid arc in G with endpoints x, y . Let $\delta_G(x_0) = \max_{p \in \gamma} \delta_G(p)$. Then there exist constants $\mu_2 = \mu_2(b, \nu, h) \geq 1$ and $\mu_3 = \mu_3(b, c, \nu, h) \geq 1$ such that*

- (1) $\text{diam}(\gamma[x, u]) \leq \mu_2 \delta_G(u)$ for $u \in \gamma[x, x_0]$, and $\text{diam}(\gamma[y, v]) \leq \mu_2 \delta_G(v)$ for $v \in \gamma[y, x_0]$;
- (2) $\text{diam}(\gamma) \leq \max \{ \mu_3 |x - y|, 2(e^h - 1) \min\{\delta_G(x), \delta_G(y)\} \}$.

Proof. We first prove (1). Obviously, it suffices to prove the first inequality in (1) because the proof for the second one is similar. Let

$$\mu_2 = \max \left\{ \frac{M_1}{q_0}, M_2 \right\},$$

where $M_1 = M_1(b, \nu, h)$ is the constant from Lemma E, $q_0 = q_0(b, \nu, h)$ and $M_2 = M_2(b, \nu, h)$ are the constants from Lemma F.

For $u \in \gamma[x, x_0]$, we divide the proof into two cases. If $\delta_G(u) \leq q_0 \delta_G(x_0)$, then Lemma F leads to

$$(2.5) \quad \text{diam}(\gamma[x, u]) \leq M_2 \delta_G(u).$$

If $\delta_G(u) > q_0 \delta_G(x_0)$, then applying Lemma E with the substitution r replaced by $\delta_G(x_0)$ and γ replaced by $\gamma[x, u]$, we easily get

$$(2.6) \quad \text{diam}(\gamma[x, u]) \leq M_1 \delta_G(x_0) < \frac{M_1}{q_0} \delta_G(u).$$

It follows from (2.5) and (2.6) that the first assertion in (1) holds, and thus the proof of (1) is complete.

To prove (2), without loss of generality, we assume that

$$\min\{\delta_G(x), \delta_G(y)\} = \delta_G(x) \quad \text{and} \quad \text{diam}(\gamma) > |x - y|.$$

Let

$$\mu_3 = \frac{3}{4} [1 + 2(1 + 6c)(e^{h+4b^2\nu \log(1+4\mu_2)} - 1)].$$

If $\delta_G(x) \geq 3c|x - y|$, then (2) follows from Lemma 1 since the constant μ_1 in Lemma 1 satisfies $\mu_1 < \mu_3$. Hence, in the following, we assume that

$$\delta_G(x) < 3c|x - y|.$$

Let $x_1 \in \gamma$ (resp. $y_1 \in \gamma$) be the first point in γ from x to y (resp. from y to x) such that

$$(2.7) \quad \text{diam}(\gamma[x, x_1]) = \frac{1}{2}|x - y| \quad (\text{resp. } \text{diam}(\gamma[y, y_1]) = \frac{1}{2}|x - y|).$$

Then we have

$$\text{diam}(\gamma[y, x_1]) \geq |y - x_1| \geq |y - x| - |x - x_1| \geq \frac{1}{2}|y - x| = \text{diam}(\gamma[x, x_1]),$$

and similarly, we get

$$\text{diam}(\gamma[x, y_1]) > \text{diam}(\gamma[y, y_1]).$$

Thus, it follows from (1) that

$$\frac{1}{2}|x - y| = \text{diam}(\gamma[x, x_1]) = \text{diam}(\gamma[y, y_1]) \leq \mu_2 \min\{\delta_G(x_1), \delta_G(y_1)\}.$$

Also,

$$|x_1 - y_1| \leq |x_1 - x| + |x - y| + |y - y_1| \leq 2|x - y|.$$

Then Lemma D implies

$$k_G(x_1, y_1) \leq 4b^2 \log \left(1 + \frac{|x_1 - y_1|}{\min\{\delta_G(x_1), \delta_G(y_1)\}} \right) \leq 4b^2 \log(1 + 4\mu_2).$$

Since γ is a (ν, h) -solid arc, for any $u_1, u_2 \in \gamma[x_1, y_1]$, we have

$$\begin{aligned} k_G(u_1, u_2) &\leq \max\{h, \ell_{k_G}(\gamma[x_1, y_1], h)\} \leq h + \nu k_G(x_1, y_1) \\ &\leq h + 4b^2 \nu \log(1 + 4\mu_2), \end{aligned}$$

and so, for all $z \in \gamma[x_1, y_1]$,

$$\begin{aligned} (2.8) \quad |z - x_1| &\leq (e^{k_G(z, x_1)} - 1) \delta_G(x_1) \\ &\leq (e^{h+4b^2\nu \log(1+4\mu_2)} - 1) (\delta_G(x) + |x - x_1|) \\ &\leq \frac{1}{2}(1 + 6c)(e^{h+4b^2\nu \log(1+4\mu_2)} - 1)|x - y|. \end{aligned}$$

Let $w_1, w_2 \in \gamma$ be points such that

$$(2.9) \quad |w_1 - w_2| \geq \frac{2}{3} \text{diam}(\gamma).$$

Then we get

Claim 2.1. $|w_1 - w_2| \leq \frac{2}{3} \mu_3 |x - y|$.

Since (2.7) guarantees that neither $\gamma[x, x_1]$ nor $\gamma[y, y_1]$ contains the set $\{w_1, w_2\}$, we see that, to prove this claim, according to the positions of w_1 and w_2 in γ , we need to consider the following four possibilities.

(1) $w_1 \in \gamma[x, x_1]$ and $w_2 \in \gamma[y, y_1]$. Obviously, by (2.7), we have

$$|w_1 - w_2| \leq |w_1 - x| + |x - y| + |y - w_2| \leq 2|x - y|.$$

(2) $w_1 \in \gamma[x, x_1]$ and $w_2 \in \gamma[x_1, y_1]$. Then (2.7) and (2.8) show that

$$|w_1 - w_2| \leq |w_1 - x_1| + |x_1 - w_2| \leq \frac{1}{2} [1 + (1 + 6c)(e^{h+4b^2\nu \log(1+4\mu_2)} - 1)] |x - y|.$$

(3) $w_1, w_2 \in \gamma[x_1, y_1]$. Then (2.8) implies

$$|w_1 - w_2| \leq |w_1 - x_1| + |x_1 - w_2| \leq (1 + 6c)(e^{h+4b^2\nu \log(1+4\mu_2)} - 1) |x - y|.$$

(4) $w_1 \in \gamma[x_1, y_1]$ and $w_2 \in \gamma[y_1, y]$. Again, we infer from (2.7) and (2.8) that

$$|w_1 - w_2| \leq |w_1 - x_1| + |x_1 - y_1| + |y_1 - w_2| \leq \frac{1}{2} [1 + 2(1 + 6c)(e^{h+4b^2\nu \log(1+4\mu_2)} - 1)] |x - y|.$$

The claim is proved.

Now, we are ready to finish the proof. It follows from (2.9) and Claim 2.1 that

$$\text{diam}(\gamma) \leq \frac{3}{2} |w_1 - w_2| \leq \mu_3 |x - y|,$$

which implies that (2) also holds in this case. Hence, the proof of the lemma is complete. \square

2.3. Quasihyperbolic mappings, coarsely quasihyperbolic mappings and relative homeomorphisms.

Definition 8. Let $G \subsetneq X$ and $G' \subsetneq Y$ be two domains. We say that a homeomorphism $f : G \rightarrow G'$ is

(1) *C-coarsely M-quasihyperbolic*, or briefly (M, C) -CQH, if there are constants $M \geq 1$ and $C \geq 0$ such that for all $x, y \in G$,

$$\frac{k_G(x, y) - C}{M} \leq k_{G'}(f(x), f(y)) \leq M k_G(x, y) + C.$$

(2) *M-quasihyperbolic*, or briefly *M-QH*, if f is $(M, 0)$ -CQH.

(3) *fully C-coarsely M-quasihyperbolic* if there are constants $M \geq 1$ and $C \geq 0$ such that f is *C-coarsely M-quasihyperbolic* in every subdomain of G .

Under coarsely quasihyperbolic mappings, we have the following useful relationship between short arcs and solid arcs.

Lemma 6. *Suppose that X and Y are rectifiable connected metric spaces, and that $G \subsetneq X$ and $G' \subsetneq Y$ are domains. If $f : G \rightarrow G'$ is (M, C) -CQH, and γ is an ε -short arc in G with $0 < \varepsilon \leq 1$, then there are constants $\nu = \nu(C, M)$ and $h = h(C, M)$ such that the image γ' of γ under f is (ν, h) -solid in G' .*

Proof. Let

$$h = (2M + 1)C + 2M \quad \text{and} \quad \nu = \frac{4(C + 1)M(M + 1)}{2C + 1}.$$

Obviously, we only need to verify that for x and $y \in \gamma$,

$$(2.10) \quad \ell_{k_{G'}}(\gamma'[x', y'], h) \leq \nu k_{G'}(x', y').$$

We prove this by considering two cases. The first case is: $k_G(x, y) < 2C + 1$. Then for $z_1, z_2 \in \gamma[x, y]$, we have

$$k_{G'}(z'_1, z'_2) \leq M k_G(z_1, z_2) + C \leq M(k_G(x, y) + \varepsilon) + C < (2M + 1)C + 2M = h,$$

and so

$$(2.11) \quad \ell_{k_{G'}}(\gamma'[x', y'], h) = 0.$$

Now, we consider the other case: $k_G(x, y) \geq 2C + 1$. Then

$$k_{G'}(x', y') \geq \frac{1}{M}(k_G(x, y) - C) > \frac{1}{2M}k_G(x, y).$$

With the aid of [37, Theorems 4.3 and 4.9], we have

$$\begin{aligned} (2.12) \quad \ell_{k_{G'}}(\gamma'[x', y'], h) &\leq \ell_{k_{G'}}(\gamma'[x', y'], (M + 1)C) \leq (M + 1)\ell_{k_G}(\gamma[x, y]) \\ &\leq (M + 1)(k_G(x, y) + \varepsilon) \\ &\leq \frac{2(C + 1)(M + 1)}{2C + 1}k_G(x, y) \\ &\leq \frac{4(C + 1)M(M + 1)}{2C + 1}k_{G'}(x', y'). \end{aligned}$$

It follows from (2.11) and (2.12) that (2.10) holds. \square

The following two lemmas are useful in the proof of Theorem 1.

Lemma 7. *Suppose that X and Y are both c -quasiconvex and complete metric spaces, and that $G \subsetneq X$ and $G' \subsetneq Y$ are domains. If both $f : G \rightarrow G'$ and $f^{-1} : G' \rightarrow G$ are weakly H -quasisymmetric, then*

- (1) f is φ -FQC, where $\varphi = \varphi_{c, H}$ which means that the function φ depends only on c and H ;
- (2) f is fully (M, C) -CQH, where $M = M(c, H) \geq 1$ and $C = C(c, H) \geq 0$ are constants.

Proof. By [25, Theorem 1.6], we know that for every subdomain $D \subset G$, both $f : D \rightarrow D'$ and $f^{-1} : D' \rightarrow D$ are φ -semisolid with $\varphi = \varphi_{c, H}$, and so, f is φ -FQC. Hence (1) holds. Meanwhile, [23, Theorem 1] implies that (1) and (2) are equivalent, and thus, (2) also holds. \square

Lemma G. ([41, Lemma 6.5]) *Suppose that X is c -quasiconvex, and that $f : X \rightarrow Y$ is weakly H -qasisymmetric. If x, y, z are distinct points in X with $|y - x| \leq t|z - x|$, then*

$$|y' - x'| \leq \theta(t)|z' - x'|,$$

where the function $\theta(t) = \theta_{c,H}(t)$ is increasing in t .

Definition 9. Let $G \subsetneq X$ and $G' \subsetneq Y$ be two domains. We say that a homeomorphism $f : G \rightarrow G'$ is

- (1) (θ, t_0) -relative if there is a constant $t_0 \in (0, 1]$ and a homeomorphism $\theta : [0, t_0] \rightarrow [0, \infty)$ such that

$$\frac{|x' - y'|}{\delta_{G'}(x')} \leq \theta\left(\frac{|x - y|}{\delta_G(x)}\right)$$

whenever $x, y \in G$ and $|x - y| < t_0 \delta_G(x)$; In particular, if $t_0 = 1$, then f is called to be θ -relative;

- (2) fully (θ, t_0) -relative (resp. fully θ -relative) if f is (θ, t_0) -relative (resp. θ -relative) in every subdomain of G .

3. THE PROOF OF THEOREM 1

In this section, we always assume that X and Y are c -quasiconvex and complete metric spaces, and that $G \subsetneq X$ and $G' \subsetneq Y$ are domains. Furthermore, we suppose that both $f : G \rightarrow G'$ and $f^{-1} : G' \rightarrow G$ are weakly H -qasisymmetric, G' is c_1 -quasiconvex and $D \subset G$ is b -uniform.

Under these assumptions, it follows from Lemma 7 that f is (M, C) -CQH with $M = M(c, H) \geq 1$ and $C = C(c, H) \geq 0$.

We are going to show the uniformity of $D' = f(D)$. For this, we let $x', y' \in D' = f(D) \subset G'$, and γ' be an ε -short arc in D' joining x' and y' with

$$0 < \varepsilon < \min \left\{ 1, \frac{1}{2} k_{D'}(x', y') \right\}.$$

Then by Lemmas 6 and 7(2), the preimage γ of γ' is a (ν, h) -solid arc in D with $\nu = \nu(c, H)$ and $h = h(c, H)$. Let $w_0 \in \gamma$ be such that

$$(3.1) \quad \delta_D(w_0) = \max_{p \in \gamma} \delta_D(p).$$

Then by Lemma 5, there is a constant $\mu = \mu(b, \nu, h)$ such that for each $u \in \gamma[x, w_0]$ and for all $z \in \gamma[u, w_0]$,

$$(3.2) \quad |u - z| \leq \text{diam}(\gamma[u, z]) \leq \mu \delta_D(z),$$

and for each $v \in \gamma[y, w_0]$ and for all $z \in \gamma[v, w_0]$,

$$|v - z| \leq \text{diam}(\gamma[v, z]) \leq \mu \delta_D(z).$$

In the following, we show that γ' is a double cone arc in D' . Precisely, we shall prove that there exist constants $A \geq 1$ and $B \geq 1$ such that for every $z' \in \gamma'[x', y']$,

$$(3.3) \quad \min\{\ell(\gamma'[x', z']), \ell(\gamma'[z', y'])\} \leq A \delta_{D'}(z')$$

and

$$(3.4) \quad \ell(\gamma') \leq B|x' - y'|.$$

The verification of (3.3) and (3.4) is given in the following two subsections.

3.1. The proof of (3.3). Let

$$A = 2e^{8b^2 A_1(C+1)M} \quad \text{and} \quad A_1 = 2e^{M+C}(1+\mu)\theta''\left(6c\theta'(\mu)e^{4b^2 M+C}\right),$$

where the functions $\theta' = \theta'_{b,H}$ and $\theta'' = \theta''_{c_1,H}$ are from Lemma G. Obviously, we only need to get the following estimate: For all $z' \in \gamma'[x', w'_0]$ (resp. $z' \in \gamma'[y', w'_0]$),

$$(3.5) \quad \ell(\gamma'[x', z']) \leq A\delta_{D'}(z') \quad (\text{resp. } \ell(\gamma'[y', z']) \leq A\delta_{D'}(z')).$$

It suffices to prove the case $z' \in \gamma'[x', w'_0]$ since the proof of the case $z' \in \gamma'[y', w'_0]$ is similar. Suppose on the contrary that there exists some point $x'_0 \in \gamma'[x', w'_0]$ such that

$$(3.6) \quad \ell(\gamma'[x', x'_0]) > A\delta_{D'}(x'_0).$$

Then we choose $x'_1 \in \gamma'[x', w'_0]$ be the first point from x' to w'_0 such that

$$(3.7) \quad \ell(\gamma'[x', x'_1]) = A\delta_{D'}(x'_1).$$

Let $x_2 \in D$ be such that

$$|x_1 - x_2| = \frac{1}{2}\delta_D(x_1).$$

Then we have

Claim 3.1. $|x'_1 - x'_2| < e^{4b^2 M+C}\delta_{D'}(x'_1).$

Obviously,

$$\delta_D(x_2) \geq \delta_D(x_1) - |x_1 - x_2| = |x_1 - x_2|,$$

and so, Lemma D implies

$$\begin{aligned} \log\left(1 + \frac{|x'_1 - x'_2|}{\delta_{D'}(x'_1)}\right) &\leq k_{D'}(x'_1, x'_2) \leq Mk_D(x_1, x_2) + C \\ &\leq 4b^2 M \log\left(1 + \frac{|x_1 - x_2|}{\min\{\delta_D(x_1), \delta_D(x_2)\}}\right) + C \\ &< 4b^2 M + C, \end{aligned}$$

whence

$$|x'_1 - x'_2| < e^{4b^2 M+C}\delta_{D'}(x'_1),$$

which shows that the claim holds.

Let $x'_3 \in \gamma'[x', x'_1]$ be such that

$$(3.8) \quad \ell(\gamma'[x', x'_3]) = \frac{1}{2}\ell(\gamma'[x', x'_1]),$$

and then, we get an estimate on $|x'_1 - x'_2|$ in terms of $d_{D'}(x'_3)$ as stated in the following claim.

Claim 3.2. $|x'_1 - x'_2| < 2e^{4b^2 M+C}\delta_{D'}(x'_3).$

It follows from (3.7) and (3.8) that

$$\delta_{D'}(x'_1) < 2\delta_{D'}(x'_3),$$

since the choice of x'_1 implies $\ell(\gamma'[x', x'_3]) < Ad_{D'}(x'_3)$. And so, Claim 3.1 leads to

$$|x'_1 - x'_2| < 2e^{4b^2M+C}\delta_{D'}(x'_3),$$

as required.

Based on Claim 3.2, we have

Claim 3.3. $|x'_1 - x'_3| \leq 2\theta'(\mu)e^{4b^2M+C}\delta_{D'}(x'_3)$.

In order to exploit Lemma G to show this claim, we need some preparation. It follows from (3.7) and (3.8) that

$$\begin{aligned} k_{D'}(x'_1, x'_3) &\geq \ell_{k_{D'}}(\gamma'[x'_1, x'_3]) - \varepsilon \geq \log \left(1 + \frac{\ell(\gamma'[x'_1, x'_3])}{\delta_{D'}(x'_1)} \right) - 1 \\ &= \log \left(1 + \frac{A}{2} \right) - 1. \end{aligned}$$

Hence, by Lemma D, we have

$$\begin{aligned} \log \left(1 + \frac{|x_1 - x_3|}{\min\{\delta_D(x_1), \delta_D(x_3)\}} \right) &\geq \frac{1}{4b^2}k_D(x_1, x_3) \geq \frac{1}{4b^2M}(k_{D'}(x'_1, x'_3) - C) \\ &\geq \frac{1}{4b^2M} \left(\log \left(1 + \frac{A}{2} \right) - 1 - C \right) \\ &> \log(1 + A_1), \end{aligned}$$

and so

$$(3.9) \quad |x_1 - x_3| > A_1 \min\{\delta_D(x_1), \delta_D(x_3)\} > \frac{A_1}{1+\mu}\delta_D(x_3),$$

since (3.2) implies

$$\delta_D(x_3) \leq \delta_D(x_1) + |x_1 - x_3| \leq (1+\mu)\delta_D(x_1).$$

Again, by (3.2), we know

$$|x_1 - x_3| \leq \mu\delta_D(x_1) = 2\mu|x_1 - x_2|.$$

Now, we are ready to apply Lemma G to the points x_1 , x_2 and x_3 in D . Since f is weakly H -quasisymmetric and D is b -uniform, by considering the restriction $f|_D$ of f onto D' , we know from Lemma G that there is an increasing function $\theta' = \theta'_{b,H}$ such that

$$|x'_1 - x'_3| \leq \theta'(2\mu)|x'_1 - x'_2|,$$

and thus, Claim 3.2 assures that

$$|x'_1 - x'_3| \leq 2\theta'(2\mu)e^{4b^2M+C}\delta_{D'}(x'_3),$$

which completes the proof of Claim 3.3.

Let us proceed the proof. To get a contradiction to the contrary assumption (3.6), we choose $x'_4 \in D'$ such that

$$(3.10) \quad |x'_3 - x'_4| = \frac{1}{3c} \delta_{D'}(x'_3).$$

Then Lemma B implies that

$$\begin{aligned} \log \left(1 + \frac{|x_3 - x_4|}{\delta_D(x_3)} \right) &\leq k_D(x_3, x_4) \leq M k_{D'}(x'_3, x'_4) + C \\ &\leq 3cM \frac{|x'_3 - x'_4|}{\delta_{D'}(x'_3)} + C \leq M + C, \end{aligned}$$

which yields that

$$(3.11) \quad |x_3 - x_4| < e^{M+C} \delta_D(x_3).$$

Meanwhile, Claim 3.3 and (3.10) imply that

$$|x'_1 - x'_3| \leq 2\theta'(2\mu)e^{4b^2M+C} \delta_{D'}(x'_3) = 6c\theta'(2\mu)e^{4b^2M+C} |x'_3 - x'_4|.$$

Now, we apply Lemma G to the points x'_1 , x'_3 and x'_4 in G' . Since $f^{-1} : G' \rightarrow G$ is weakly H -quasisymmetric and G' is c_1 -quasiconvex, we know from Lemma G that there is an increasing function $\theta'' = \theta''_{c_1, H}$ such that

$$|x_1 - x_3| \leq \theta'' \left(6c\theta'(2\mu)e^{4b^2M+C} \right) |x_3 - x_4|,$$

which, together with (3.9) and (3.11), shows that

$$\begin{aligned} |x_1 - x_3| &\leq e^{M+C} \theta'' \left(6c\theta'(2\mu)e^{4b^2M+C} \right) \delta_D(x_3) \\ &\leq \frac{1+\mu}{A_1} e^{M+C} \theta'' \left(6c\theta'(2\mu)e^{4b^2M+C} \right) |x_1 - x_3| \\ &= \frac{1}{2} |x_1 - x_3|. \end{aligned}$$

This obvious contradiction shows that (3.3) is true. \square

3.2. The proof of (3.4). Let

$$B = 12cA^2 e^{6b^2M\mu \left(1 + \theta'' \left(\frac{1+12cA}{3c} \right) \right)},$$

and suppose on the contrary that

$$(3.12) \quad \ell(\gamma') > B|x' - y'|.$$

Since $\frac{9}{2}ce^{\frac{3}{2}} < B$, we see from Lemma 2 that

$$(3.13) \quad |x' - y'| > \frac{1}{3c} \max\{\delta_{D'}(x'), \delta_{D'}(y')\}.$$

For convenience, in the following, we assume that

$$\max\{\delta_{D'}(x'), \delta_{D'}(y')\} = \delta_{D'}(x').$$

First, we choose some special points from γ' . By (3.12), we know that there exist w'_1 and $w'_2 \in \gamma'$ such that x' , w'_1 , w'_2 and y' are successive points in γ' and

$$(3.14) \quad \ell(\gamma'[x', w'_1]) = \ell(\gamma'[w'_2, y']) = 6cA|x' - y'|.$$

Then we have

Claim 3.4. $|x' - w'_1| \geq \frac{1}{2}\delta_{D'}(w'_1)$ and $|y' - w'_2| \geq \frac{1}{2}\delta_{D'}(w'_2)$.

Obviously, it suffices to show the first inequality in the claim. Suppose

$$|x' - w'_1| < \frac{1}{2}\delta_{D'}(w'_1).$$

Then (3.3) and (3.13) lead to

$$\delta_{D'}(x') \geq \delta_{D'}(w'_1) - |x' - w'_1| > \frac{1}{2}\delta_{D'}(w'_1) \geq \frac{1}{2A}\ell(\gamma'[x', w'_1]) = 3c|x' - y'| > \delta_{D'}(x').$$

This obvious contradiction completes the proof of Claim 3.4.

By using Claim 3.4, we get a lower bound for $|w_1 - w_2|$ in terms of $\min\{\delta_D(w_1), \delta_D(w_2)\}$, which is as follows.

Claim 3.5. $|w_1 - w_2| > \left(1 + \theta''\left(\frac{1+12cA}{3c}\right)\right)\mu \min\{\delta_D(w_1), \delta_D(w_2)\}.$

Without loss of generality, we assume that $\min\{\delta_D(w_1), \delta_D(w_2)\} = \delta_D(w_1)$. Then by (3.14) and Claim 3.4, we have

$$(3.15) \quad \delta_{D'}(w'_1) \leq 2|x' - w'_1| \leq 2\ell(\gamma'[x', w'_1]) = 12cA|x' - y'|.$$

Since γ' is an ε -short arc and D is b -uniform, by Lemma D, we have

$$\begin{aligned} \log\left(1 + \frac{|w_1 - w_2|}{\delta_D(w_1)}\right) &\geq \frac{1}{4b^2}k_D(w_1, w_2) \geq \frac{1}{4Mb^2}k_{D'}(w'_1, w'_2) - \frac{C}{4Mb^2} \\ &\geq \frac{1}{4Mb^2}\ell_{k_{D'}}(\gamma'[w'_1, w'_2]) - \frac{\varepsilon + C}{4Mb^2} \\ &\geq \frac{1}{4Mb^2}\log\left(1 + \frac{\ell(\gamma'[w'_1, w'_2])}{\delta_{D'}(w'_1)}\right) - \frac{1 + C}{4Mb^2} \\ &\geq \frac{1}{4Mb^2}\log\left(1 + \frac{B - 12cA}{12cA}\right) - \frac{1 + C}{4Mb^2} \\ &= \lambda, \end{aligned}$$

where the last inequality follows from (3.15) and the following inequalities:

$$\ell(\gamma'[w'_1, w'_2]) = \ell(\gamma') - \ell(\gamma'[x', w'_1]) - \ell(\gamma'[y', w'_2]) > (B - 12cA)|x' - y'|.$$

Hence

$$|w_1 - w_2| \geq (e^\lambda - 1)\delta_D(w_1) > \left(1 + \theta''\left(\frac{1 + 12cA}{3c}\right)\right)\mu\delta_D(w_1),$$

as required.

Next, we get the following upper bound for $|w_1 - w_2|$ in terms of $\min\{\delta_D(w_1), \delta_D(w_2)\}$.

Claim 3.6. $|w_1 - w_2| \leq \theta''\left(\frac{1+12cA}{3c}\right)\mu \min\{\delta_D(w_1), \delta_D(w_2)\}.$

First, we see that $w_0 \in \gamma[w_1, y]$, where w_0 is the point in γ which satisfies (3.1), because otherwise (3.2) gives that

$$|w_1 - w_2| \leq \mu \delta_D(w_1),$$

which contradicts with Claim 3.5.

We are going to apply Lemma G to the points x' , w'_1 and w'_2 in G' . We need a relationship between $|w'_1 - w'_2|$ and $|x' - w'_1|$. To this end, it follows from (3.14) that

$$|w'_1 - w'_2| \leq |w'_1 - x'| + |x' - y'| + |y' - w'_2| \leq (1 + 12cA)|x' - y'| \leq \frac{1 + 12cA}{3c}|x' - w'_1|,$$

since we infer from the choice of w'_1 , (3.3) and Claim 3.4 that

$$|x' - w'_1| \geq \frac{1}{2}\delta_{D'}(w'_1) \geq \frac{1}{2A}\ell(\gamma'[x', w'_1]) = 3c|x' - y'|.$$

Then by Lemma G, we have known that there is an increasing function $\theta'' = \theta''_{c_1, H}$ such that

$$|w_1 - w_2| \leq \theta'' \left(\frac{1 + 12cA}{3c} \right) |x - w_1|,$$

and thus, (3.2) leads to

$$|w_1 - w_2| \leq \theta'' \left(\frac{1 + 12cA}{3c} \right) \mu \min\{\delta_D(w_1), \delta_D(w_2)\},$$

which shows that Claim 3.6 holds.

It follows from Claims 3.5 and 3.6 that it is impossible, and so this obvious contradiction completes the proof of (3.4). \square

Inequalities (3.3) and (3.4), together with the arbitrariness of the choice of x' and y' in D' , show that D' is B -uniform, which implies that Theorem 1 holds. \square

4. THE PROOF OF THEOREM 2

In this section, we always assume that X and Y are both c -quasiconvex and complete metric spaces, that $G \subsetneq X$ is a non-point-cut and locally a -John domain, and that $G' \subsetneq Y$ is a b_1 -uniform and locally a -John domain. Further, we assume that $f : G \rightarrow G'$ is a φ -FQC mapping and D is a b_2 -uniform subdomain of G .

We divide this section into two subsections. In the first subsection, a useful lemma will be proved, and the proof of Theorem 2 will be presented in the second subsection.

4.1. An auxiliary result. First, based on [23, Theorems 1 and 2], we prove the following result which plays a key role in the proof of Theorem 2.

Lemma 8. *Under the given assumptions in the first paragraph of this section, we have the following assertions.*

- (1) *There exist constants $M = M(c, \varphi) \geq 1$ and $C = C(c, \varphi) \geq 0$ such that both $f : G \rightarrow G'$ and $f^{-1} : G' \rightarrow G$ are fully (M, C) -CQH.*
- (2) *There exists a constant $q = q(c) \in (0, 1)$ such that for any $x \in G$, f is η -quasisymmetric in $\mathbb{B}(x, q\delta_G(x))$ and f^{-1} is η -quasisymmetric in $\mathbb{B}(x', q\delta_{G'}(x'))$, where $\eta = \eta_{a, c, \varphi}$.*

Proof. Since f is a φ -FQC mapping, we see that Lemma 8(1) easily follows from [23, Theorem 1]. For the proof of the second assertion, we infer from [23, Theorem 2] that we only need to prove that there is a homeomorphism $\theta : [0, 1) \rightarrow [0, \infty)$ such that both f and f^{-1} are θ -relative. By symmetry, we know that we only need to show that f is θ -relative. To this end, we let $0 < t < 1$ and $x, y \in G$ with $|x - y| = t\delta_G(x)$. Then we separate the proof into two cases. For the first case, that is, $0 < t \leq \frac{1}{3c}$, it follows from Lemma B that

$$k_G(x, y) \leq 3c \frac{|x - y|}{\delta_G(x)} = 3ct.$$

Hence,

$$(4.1) \quad \frac{|x' - y'|}{\delta_{G'}(x')} \leq e^{k_{G'}(x', y')} - 1 \leq e^{\varphi(k_G(x, y))} - 1 \leq e^{\varphi(3ct)} - 1.$$

For the other case, that is, $\frac{1}{3c} < t < 1$, by Lemma 4, we know that

$$k_G(x, y) \leq 2a \frac{3+t}{1-t},$$

which implies

$$(4.2) \quad \frac{|x' - y'|}{\delta_{G'}(x')} \leq e^{k_{G'}(x', y')} - 1 \leq e^{\varphi(k_G(x, y))} - 1 \leq e^{\varphi(2a \frac{3+t}{1-t})} - 1.$$

Therefore, (4.1) and (4.2) show that f is θ -relative, where

$$\theta(t) = \begin{cases} e^{\varphi(\frac{6ac(9c+1)}{3c-1}t)} - 1, & \text{if } t \in [0, \frac{1}{3c}], \\ e^{\varphi(2a \frac{3+t}{1-t})} - 1, & \text{if } t \in (\frac{1}{3c}, 1). \end{cases}$$

Hence the proof of Lemma 8 is complete. \square

4.2. The proof of Theorem 2. For $x', y' \in D'$, there is an ε -short arc γ' joining x' and y' in D' with

$$0 < \varepsilon < \frac{1}{2} \min\{2, k_{D'}(x', y')\}.$$

Lemmas 6 and 8(1) show that the preimage γ of γ' is a (ν, h) -solid arc in D , where $\nu = \nu(c, \varphi)$, $h = h(c, \varphi)$. Let $z_0 \in \gamma$ be such that

$$\delta_D(z_0) = \max_{p \in \gamma} \delta_D(p).$$

Then by Lemma 5, there is a constant $\mu = \mu(b_2, \nu, h) = \mu(b_2, c, \varphi) > 1$ such that for each $u \in \gamma[x, z_0]$ and for all $z \in \gamma[u, z_0]$,

$$(4.3) \quad |u - z| \leq \mu \delta_D(z),$$

and for each $v \in \gamma[y, z_0]$ and for all $z \in \gamma[v, z_0]$,

$$(4.4) \quad |v - z| \leq \mu \delta_D(z).$$

In the following, we show that γ' is a double cone arc in D' . The proof is divided into two steps which are given in the following two subsubsections.

4.2.1. *The verification of the cigar condition.* To this end, we let

$$\lambda_1 = \lambda_2^2(1 + \lambda_2^4)^{9(C+1)M^2b_2^2\lambda_2}$$

and

$$\lambda_2 = \max \left\{ \frac{3c\mu}{\varphi^{-1}(\frac{q}{21ce})}, \frac{12c(C+1)\mu}{q}, 16b_1^2b_2^2(C+1)M^2, \frac{2(1+\mu)}{\eta^{-1}(\frac{1}{3}\eta^{-1}(\frac{1}{\mu}))} \right\},$$

where the constants C , M , q and the function η are from Lemma 8.

In this subsubsection, we show that γ' satisfies the cigar condition with constant $2\lambda_1^2$. Obviously, we may assume that

(1) either there exists a point $z'_1 \in \gamma'[x', z'_0]$ such that

$$\ell(\gamma'[x', z'_1]) > \lambda_1 \delta_{D'}(z'_1),$$

(2) or there exists a point $z'_2 \in \gamma'[y', z'_0]$ such that

$$\ell(\gamma'[y', z'_2]) > \lambda_1 \delta_{D'}(z'_2).$$

If (1) happens, then we let $x'_0 \in \gamma'[x', z'_0]$ be the first point from x' to z'_0 such that

$$(4.5) \quad \ell(\gamma'[x', x'_0]) = \lambda_1 \delta_{D'}(x'_0).$$

The following comparison result is useful.

Claim 4.1. For every $z \in \gamma[x_0, z_0]$, $\delta_G(z) \leq \lambda_2 \delta_D(z)$.

Suppose on the contrary that there exists some point $x_1 \in \gamma[x_0, z_0]$ with

$$(4.6) \quad \delta_G(x_1) > \lambda_2 \delta_D(x_1).$$

To get a contradiction, we need some preparation. First, it follows from (4.3) that for all $z \in \gamma[x, x_1]$,

$$|z - x_1| \leq \mu \delta_D(x_1) < \frac{\mu}{\lambda_2} \delta_G(x_1) \leq \frac{q}{12c(C+1)} \delta_G(x_1),$$

whence

$$(4.7) \quad \gamma[x, x_1] \subset \mathbb{B}(x_1, \frac{\mu}{\lambda_2} \delta_G(x_1)) \subset \mathbb{B}(x_1, \frac{q}{12c(C+1)} \delta_G(x_1)).$$

Further, we show that

$$(4.8) \quad \gamma'[x', x'_1] \subset \mathbb{B}(x'_0, \frac{q}{10c} \delta_{G'}(x'_0)).$$

For each $z' \in \gamma'[x', x'_1]$, by Lemma B, we have

$$k_{G'}(z', x'_1) \leq \varphi(k_G(z, x_1)) \leq \varphi(3c \frac{|x_1 - z|}{\delta_G(x_1)}) \leq \varphi(\frac{3c\mu}{\lambda_2}) \leq \frac{q}{21ce},$$

and so by the elementary inequality “ $e^x - 1 \leq ex$ ” in $(0, 1)$, we get that

$$|z' - x'_1| \leq (e^{k_{G'}(z', x'_1)} - 1) \delta_{G'}(x'_1) \leq \frac{q}{21c} \delta_{G'}(x'_1),$$

which implies

$$\delta_{G'}(x'_0) \geq \delta_{G'}(x'_1) - |x'_0 - x'_1| \geq \frac{19}{20c} \delta_{G'}(x'_1).$$

Hence

$$|z' - x'_0| \leq |z' - x'_1| + |x'_0 - x'_1| \leq \frac{2q}{21c} \delta_{G'}(x'_1) < \frac{q}{10c} \delta_{G'}(x'_0),$$

as required.

Next, we need to choose some special points. Let $x'_3 \in \gamma'[x', x'_0]$ be such that

$$\ell(\gamma'[x', x'_3]) = \frac{1}{2} \ell(\gamma'[x', x'_0]),$$

and let $x'_2 \in \partial D'$ satisfy

$$|x'_2 - x'_0| \leq \frac{3}{2} \delta_{D'}(x'_0).$$

Then we assert that

$$(4.9) \quad x'_2, x'_3 \in \mathbb{B}(x'_0, q\delta_{G'}(x'_0)).$$

It follows from (4.8) that we only need to verify the truth of $x'_2 \in \mathbb{B}(x'_0, q\delta_{G'}(x'_0))$. Obviously, (4.5) shows that

$$(4.10) \quad |x'_2 - x'_0| \leq \frac{3}{2} \delta_{D'}(x'_0) = \frac{3}{\lambda_1} \ell(\gamma'[x', x'_3]) < 3\delta_{D'}(x'_3),$$

since the choice of x'_0 implies that $\ell(\gamma'[x', x'_3]) < \lambda_1 \delta_{D'}(x'_3)$. Also, we have

$$(4.11) \quad \begin{aligned} k_{D'}(x'_0, x'_3) &\geq \ell_{k_{D'}}(\gamma'[x'_0, x'_3]) - \varepsilon \geq \log \left(1 + \frac{\ell(\gamma'[x'_0, x'_3])}{\delta_{D'}(x'_0)} \right) - 1 \\ &= \log \left(1 + \frac{\lambda_1}{2} \right) - 1 > 1, \end{aligned}$$

whence

$$|x'_0 - x'_3| > \frac{1}{3c} \delta_{D'}(x'_0),$$

because otherwise, by Lemma B, we get

$$k_{D'}(x'_0, x'_3) < \frac{1}{3c} \frac{|x'_0 - x'_3|}{\delta_{D'}(x'_0)} \leq \frac{1}{9c^2} < 1,$$

which contradicts with (4.11). Hence we deduce from (4.8) and the choice of x'_2 that

$$(4.12) \quad |x'_2 - x'_0| \leq \frac{3}{2} \delta_{D'}(x'_0) \leq \frac{9c}{2} |x'_0 - x'_3| \leq \frac{9q}{20} \delta_{G'}(x'_0),$$

as needed.

It is (4.9) that allows us to apply Lemma 8(2) to the points x'_0, x'_2 and x'_3 , which shows

$$\frac{|x_2 - x_0|}{|x_3 - x_0|} \leq \eta \left(\frac{|x'_2 - x'_0|}{|x'_3 - x'_0|} \right),$$

and thus, (4.10) implies

$$(4.13) \quad |x'_3 - x'_0| \leq \frac{1}{\eta^{-1}(\frac{1}{\mu})} |x'_2 - x'_0| < \frac{3}{\eta^{-1}(\frac{1}{\mu})} \delta_{D'}(x'_3),$$

since (4.3) and the choice of x'_2 assure that

$$|x_3 - x_0| \leq \mu \delta_D(x_0) \leq \mu |x_2 - x_0|.$$

Still, we need an estimate on $|x_3 - x_0|$. Since it follows from Lemma D and (4.11) that

$$\begin{aligned} \log \left(1 + \frac{|x_3 - x_0|}{\min\{\delta_D(x_0), \delta_D(x_3)\}} \right) &\geq \frac{1}{4b_2^2} k_D(x_0, x_3) \geq \frac{1}{4b_2^2 M} (k_{D'}(x'_0, x'_3) - C) \\ &\geq \frac{1}{4b_2^2 M} \left(\log \left(1 + \frac{\lambda_1}{2} \right) - 1 - C \right) \\ &> \log(1 + \lambda_2), \end{aligned}$$

we know

$$(4.14) \quad |x_3 - x_0| > \lambda_2 \min\{\delta_D(x_0), \delta_D(x_3)\} > \frac{\lambda_2}{1 + \mu} \delta_D(x_3),$$

since by (4.3), $\delta_D(x_3) \leq \delta_D(x_0) + |x_3 - x_0| \leq (1 + \mu) \delta_D(x_0)$.

Now, we are ready to get a contradiction to the contrary assumption (4.6). Let $x_4 \in \partial D$ be such that

$$|x_4 - x_3| \leq 2\delta_D(x_3).$$

Then again by (4.3) and the contrary assumption (4.6), we know that

$$\begin{aligned} |x_4 - x_1| &\leq |x_4 - x_3| + |x_3 - x_1| \leq 2\delta_D(x_3) + |x_3 - x_1| \leq 2\delta_D(x_1) + 3|x_3 - x_1| \\ &\leq (2 + 3\mu)\delta_D(x_1) < \frac{2 + 3\mu}{\lambda_2} \delta_G(x_1) \\ &< \frac{q}{2} \delta_G(x_1), \end{aligned}$$

which, together with (4.7), implies

$$x_0, x_3, x_4 \in \mathbb{B}(x_1, \frac{q}{2} \delta_G(x_1)).$$

Apply Lemma 8(2) to the points x_0 , x_3 , and x_4 . Then we see from (4.13), (4.14) and the choice of x_4 that

$$\frac{1}{3} \eta^{-1} \left(\frac{1}{\mu} \right) \leq \frac{|x'_4 - x'_3|}{|x'_3 - x'_0|} \leq \eta \left(\frac{|x_4 - x_3|}{|x_3 - x_0|} \right) < \eta \left(\frac{2(1 + \mu)}{\lambda_2} \right),$$

which is the desired contradiction since $\lambda_2 \geq \frac{2(1 + \mu)}{\eta^{-1} \left(\frac{1}{3} \eta^{-1} \left(\frac{1}{\mu} \right) \right)}$. Hence Claim 4.1 holds.

If (2) happens, then we let y'_0 be the first point in $\gamma'[y', z'_0]$ from y' to z'_0 such that

$$\gamma'[y', y'_0] = \lambda_1 \delta_{D'}(y'_0),$$

and the similar reasoning as in the proof of Claim 4.1 implies

Claim 4.2. For every $z \in \gamma[y_0, z_0]$, we have $\delta_G(z) \leq \lambda_2 \delta_D(z)$.

In order to prove that γ' satisfies the cigar condition with constant $2\lambda_1^2$, we only need to consider the case where both (1) and (2) happen because the proofs for other cases are similar, and in fact, the corresponding discussions are simpler. First, we partition the part $\gamma'[x'_0, y'_0]$ of γ' as follows. Let $u'_0 \in \gamma'[x'_0, y'_0]$ be such that

$$\delta_{D'}(u'_0) = \max_{p' \in \gamma'[x'_0, y'_0]} \delta_{D'}(p').$$

Obviously, there exists a unique integer $k \geq 0$ with

$$2^k \delta_{D'}(x'_0) \leq \delta_{D'}(u'_0) < 2^{k+1} \delta_{D'}(x'_0).$$

Let $v'_0 = x'_0$. If $k = 0$, then we let $v'_1 = u'_0$. If $k > 1$, then for each $i \in \{1, \dots, k\}$, we let v'_i be the first point in $\gamma'[x'_0, u'_0]$ from v'_{i-1} to u'_0 such that

$$\delta_{D'}(v'_i) = 2^i \delta_{D'}(x'_0),$$

and let $v'_{k+1} = u'_0$. It is possible that $v'_{k+1} = v'_k$. This possibility happens if and only if $\delta_{D'}(u'_0) = 2^k \delta_{D'}(x'_0)$.

As for this partition, we have the following assertion.

Claim 4.3. For each $i \in \{0, \dots, k\}$, we have

- (1) $\ell(\gamma'[v'_i, v'_{i+1}]) \leq \lambda_2^4 \delta_{D'}(v'_i)$, and
- (2) for $z' \in \gamma'[v'_i, v'_{i+1}]$,

$$\delta_{D'}(v'_i) < (1 + \lambda_2^4)^{20b_1^2 b_2^2 M^2 \lambda_2} \delta_{D'}(z').$$

We first prove (1). It follows from Lemmas D and 8(1) that

$$\begin{aligned} \ell_{k_{D'}}(\gamma'[v'_i, v'_{i+1}]) &\leq k_{D'}(v'_i, v'_{i+1}) + \varepsilon \leq M k_D(v_i, v_{i+1}) + C + 1 \\ &\leq 4b_2^2 M \log \left(1 + \frac{|v_i - v_{i+1}|}{\min\{\delta_D(v_i), \delta_D(v_{i+1})\}} \right) + C + 1, \end{aligned}$$

and thus, Claims 4.1 and 4.2, together with Lemma 8(1), lead to

$$\begin{aligned} \ell_{k_{D'}}(\gamma'[v'_i, v'_{i+1}]) &\leq 4b_2^2 M \log \left(1 + \frac{\lambda_2 |v_i - v_{i+1}|}{\min\{\delta_G(v_i), \delta_G(v_{i+1})\}} \right) + C + 1 \\ &\leq 4b_2^2 M \lambda_2 k_G(v_i, v_{i+1}) + C + 1 \\ &\leq 4b_2^2 M^2 \lambda_2 k_{G'}(v'_i, v'_{i+1}) + 4b_2^2 C M \lambda_2 + C + 1, \end{aligned}$$

and finally, with the aid of Lemma D, we obtain

$$\begin{aligned} (4.15) \quad \ell_{k_{D'}}(\gamma'[v'_i, v'_{i+1}]) &\leq 16b_1^2 b_2^2 M^2 \lambda_2 \log \left(1 + \frac{|v'_i - v'_{i+1}|}{\min\{\delta_{G'}(v'_i), \delta_{G'}(v'_{i+1})\}} \right) \\ &\quad + 4b_2^2 C M \lambda_2 + C + 1 \\ &< 16b_1^2 b_2^2 M^2 \lambda_2 \log \left(1 + \frac{\ell(\gamma'[v'_i, v'_{i+1}])}{\delta_{D'}(v'_i)} \right) + 4b_2^2 C M \lambda_2 \\ &\quad + C + 1. \end{aligned}$$

Meanwhile, the choice of v'_i gives

$$\ell_{k_{D'}}(\gamma'[v'_i, v'_{i+1}]) = \int_{\gamma'[v'_i, v'_{i+1}]} \frac{|dz|}{\delta_{D'}(z)} \geq \frac{\ell(\gamma'[v'_i, v'_{i+1}])}{2\delta_{D'}(v'_i)},$$

and so,

$$\ell(\gamma'[v'_i, v'_{i+1}]) \leq \lambda_2^4 \delta_{D'}(v'_i).$$

Thus Claim 4.3(1) holds.

Moreover, by (4.15) and Claim 4.3(1), we see that for all $z' \in \gamma'[v'_i, v'_{i+1}]$,

$$\begin{aligned} \log \frac{\delta_{D'}(v'_i)}{\delta_{D'}(z')} &\leq k_{D'}(v'_i, z') \leq \ell_{k_{D'}}(\gamma'[v'_i, v'_{i+1}]) \\ &\leq 16b_1^2 b_2^2 M^2 \lambda_2 \log(1 + \lambda_2^4) + 4b_2^2 CM \lambda_2 + C + 1 \\ &\leq 20b_1^2 b_2^2 M^2 \lambda_2 \log(1 + \lambda_2^4). \end{aligned}$$

Hence

$$\delta_{D'}(v'_i) < (1 + \lambda_2^4)^{20b_1^2 b_2^2 M^2 \lambda_2} \delta_{D'}(z'),$$

which shows that Claim 4.3(2) is true too, and thus the claim is proved.

Now, we are ready to verify that γ' satisfies the $2\lambda_1^2$ -cigar condition, that is, for every $z' \in \gamma'[x', y']$,

$$(4.16) \quad \min\{\ell(\gamma'[x', z']), \ell(\gamma'[z', y'])\} \leq 2\lambda_1^2 \delta_{D'}(z').$$

Obviously, we only need to consider the case $z' \in \gamma'[x', u'_0]$ since the proof for the case $z' \in \gamma'[y', u'_0]$ is similar.

If $z' \in \gamma'[x', x'_0]$, then (4.16) immediately follows from the choice of x'_0 .

If $z' \in \gamma'[x'_0, u'_0]$, then there must exist a $j \in \{0, \dots, k\}$ such that

$$z' \in \gamma'[v'_j, v'_{j+1}],$$

and so by (4.5) and Claim 4.3, we have the following:

$$\begin{aligned} \ell(\gamma'[x', z']) &= \ell(\gamma'[x', x'_0]) + \ell(\gamma'[x'_0, z']) \leq \lambda_1 \delta_{D'}(x'_0) + \sum_{i=0}^j \ell(\gamma'[v'_i, v'_{i+1}]) \\ &\leq \lambda_1 \delta_{D'}(v'_j) + \lambda_2^4 \sum_{i=0}^j \delta_{D'}(v'_i) \leq (\lambda_1 + 2\lambda_2^4) \delta_{D'}(v'_j) \\ &< 2\lambda_1^2 \delta_{D'}(z'), \end{aligned}$$

which shows that (4.16) is also true in this case. Hence γ' satisfies the $2\lambda_1^2$ -cigar condition. \square

4.2.2. *The verification of the turning condition.* We only need to prove that

$$(4.17) \quad \ell(\gamma') \leq \lambda_0 |x' - y'|,$$

where

$$\lambda_0 = \max \left\{ 24c\lambda_1^2 e^{1+C} e^{6Mb_2^2[1+\mu\eta(8\lambda_1^2+\frac{1}{3c})]}, 48c\lambda_1^2 e^\rho \right\}$$

and

$$\rho = \frac{96(c+1)(b_1 b_2 M)^2}{q\varphi^{-1}(\log \frac{3}{2})\eta^{-1}(\frac{q}{12\lambda_1^2})} \log \left(1 + 4c\lambda_1^2 + \frac{1}{6c} \right) + \frac{24(c+1)MCb_2^2}{q\varphi^{-1}(\log \frac{3}{2})\eta^{-1}(\frac{q}{12\lambda_1^2})} + C + 1.$$

We prove (4.17) by contradiction. Suppose that

$$(4.18) \quad \ell(\gamma') > \lambda_0 |x' - y'|.$$

Then Lemma 2 implies

$$|x' - y'| > \frac{1}{3c} \max\{\delta_{D'}(x'), \delta_{D'}(y')\},$$

and also, we know that there exist $w'_1, w'_2 \in \gamma'$ such that x', w'_1, w'_2 and y' are successive pints in γ' , and

$$(4.19) \quad \ell(\gamma'[x', w'_1]) = \ell(\gamma'[w'_2, y']) = 12c\lambda_1^2|x' - y'|.$$

As for this partition of γ' , we prove several claims.

Claim 4.4. $|x' - w'_1| \geq \frac{1}{2}\delta_{D'}(w'_1)$ and $|y' - w'_2| \geq \frac{1}{2}\delta_{D'}(w'_2)$.

The proof of this claim easily follows from a similar argument as in the proof of Claim 3.4 with the substitution A by $2\lambda_1^2$.

Claim 4.5. $6c|x' - y'| \leq \min\{\delta_{D'}(w'_1), \delta_{D'}(w'_2)\} \leq \max\{\delta_{D'}(w'_1), \delta_{D'}(w'_2)\} \leq 24c\lambda_1^2|x' - y'|$.

It is equivalent to show that $6c|x' - y'| \leq \delta_{D'}(w'_i) \leq 24c\lambda_1^2|x' - y'|$ for $i \in \{1, 2\}$, which easily follows from Claim 4.4, (4.16) and (4.19).

Claim 4.6. $|w_1 - w_2| > \mu\eta(8\lambda_1^2 + \frac{1}{3c}) \min\{\delta_D(w_1), \delta_D(w_2)\}$.

For convenience, we may assume that

$$\min\{\delta_D(w_1), \delta_D(w_2)\} = \delta_D(w_1).$$

Since D is b_2 -uniform, we see from Lemma D that

$$\log\left(1 + \frac{|w_1 - w_2|}{\delta_D(w_1)}\right) \geq \frac{1}{4b_2^2}k_D(w_1, w_2) \geq \frac{1}{4Mb_2^2}(k_{D'}(w'_1, w'_2) - C),$$

and thus, the assumption “ γ' being an ε -short arc” implies

$$\begin{aligned} \log\left(1 + \frac{|w_1 - w_2|}{\delta_D(w_1)}\right) &\geq \frac{1}{4Mb_2^2}(\ell_{k_{D'}}(\gamma'[w'_1, w'_2]) - \varepsilon - C) \\ &\geq \frac{1}{4Mb_2^2}\left(\log\left(1 + \frac{\ell(\gamma'[w'_1, w'_2])}{\delta_{D'}(w'_1)}\right) - 1 - C\right) \\ &\geq \frac{1}{4Mb_2^2}\left(\log\left(1 + \frac{\lambda_0 - 24c\lambda_1^2}{24c\lambda_1^2}\right) - 1 - C\right) \\ &= \mu_1, \end{aligned}$$

where the last inequality holds because of Claim 4.5 and the following chain of inequalities which are from (4.18) and (4.19):

$$(4.20) \quad \ell(\gamma'[w'_1, w'_2]) = \ell(\gamma') - \ell(\gamma'[x', w'_1]) - \ell(\gamma'[y', w'_2]) > (\lambda_0 - 24c\lambda_1^2)|x' - y'|.$$

Hence

$$|w_1 - w_2| \geq (e^{\mu_1} - 1)\delta_D(w_1) > \mu\eta\left(8\lambda_1^2 + \frac{1}{3c}\right)\delta_D(w_1),$$

as needed.

With the aid of (4.3), the following is a direct consequence of Claim 4.6.

Claim 4.7. $|x - w_1| \leq \mu\delta_D(w_1)$.

For points in $\gamma'[w'_1, w'_2]$, we have the following comparison result.

Claim 4.8. For all $w' \in \gamma'[w'_1, w'_2]$, $\delta_{G'}(w') \leq \frac{12\lambda_1^2}{q}\delta_{D'}(w')$.

Suppose that there exists some point $w'_3 \in \gamma'[w'_1, w'_2]$ with

$$(4.21) \quad \delta_{G'}(w'_3) > \frac{12\lambda_1^2}{q}\delta_{D'}(w'_3).$$

Without loss of generality, we may assume that

$$\min\{\ell(\gamma'[x', w'_3]), \ell(\gamma'[y', w'_3])\} = \ell(\gamma'[x', w'_3]),$$

since the proof for the other case is similar.

By (4.16) and (4.21), for all $w' \in \gamma'[x', w'_3]$, we have

$$\ell(\gamma'[w', w'_3]) \leq 2\lambda_1^2\delta_{D'}(w'_3) < \frac{q}{6}\delta_{G'}(w'_3),$$

and so, (4.19) gives

$$\begin{aligned} |w'_2 - w'_3| &\leq |w'_2 - y'| + |y' - x'| + |x' - w'_3| \\ &\leq \ell(\gamma'[y', w'_2]) + \frac{1}{12c\lambda_1^2}\ell(\gamma'[x', w'_1]) + \ell(\gamma'[x', w'_3]) \\ &\leq \left(2 + \frac{1}{12c\lambda_1^2}\right)\ell(\gamma'[x', w'_3]) < \frac{q}{2}\delta_{G'}(w'_3). \end{aligned}$$

Then we know that

$$x', w'_1, w'_2 \in \mathbb{B}(w'_3, \frac{q}{2}\delta_{G'}(w'_3)).$$

By applying Lemma 8(2) to the points x' , w'_1 and w'_2 , together with (4.21) and Claims 4.4 ~ 4.7, we know that

$$\eta\left(8\lambda_1^2 + \frac{1}{3c}\right) < \frac{|w_2 - w_1|}{|x - w_1|} \leq \eta\left(\frac{|w'_1 - w'_2|}{|x' - w'_1|}\right) \leq \eta\left(8\lambda_1^2 + \frac{1}{3c}\right),$$

since by (4.19),

$$|w'_1 - w'_2| \leq |w'_2 - y'| + |y' - x'| + |x' - w'_1| \leq (1 + 24c\lambda_1^2)|x' - y'|.$$

This is the desired contradiction, from which the claim follows.

The next result is an analogue of Claim 4.8 for points in $\gamma[w_1, w_2]$.

Claim 4.9. For all $w \in \gamma[w_1, w_2]$, we have

$$\delta_G(w) \leq \frac{6(c+1)}{q\varphi^{-1}(\log \frac{3}{2})\eta^{-1}(\frac{q}{12\lambda_1^2})}\delta_D(w).$$

Suppose that there exists some point $u \in \gamma[w_1, w_2]$ such that

$$\delta_G(u) > \frac{6(c+1)}{q\varphi^{-1}(\log \frac{3}{2})\eta^{-1}(\frac{q}{12\lambda_1^2})}\delta_D(u).$$

Let

$$u_1 \in \mathbb{S}\left(u, \frac{q\varphi^{-1}(\log \frac{3}{2})}{3(c+1)}\delta_G(u)\right).$$

Then Lemma 3 guarantees that $u_1 \in G$, and thus, Lemma B implies

$$k_{G'}(u'_1, u') \leq \varphi(k_G(u_1, u)) \leq \varphi\left(3c \frac{|u_1 - u|}{\delta_G(u)}\right) < \log \frac{3}{2},$$

which leads to

$$(4.22) \quad |u'_1 - u'| \leq \frac{1}{2} \delta_{G'}(u').$$

Let $u_2 \in \partial D$ be such that

$$|u_2 - u| \leq 2\delta_D(u).$$

Thus

$$(4.23) \quad |u_2 - u| \leq 2\delta_D(u) \leq \frac{q\varphi^{-1}(\log \frac{3}{2})}{3(c+1)} \eta^{-1}\left(\frac{q}{12\lambda_1^2}\right) \delta_G(u) < \frac{q}{3(c+1)} \delta_G(u).$$

Hence, the choice of u_1 implies

$$u_1, u_2 \in \mathbb{B}(u, \frac{q}{2} \delta_G(u)) \cap G.$$

Apply Lemma 8(2) to the points u, u_1 and u_2 . Then (4.22) and (4.23) lead to

$$\frac{q}{6\lambda_1^2} < \frac{|u'_2 - u'|}{|u'_1 - u'|} \leq \eta\left(\frac{|u_2 - u|}{|u_1 - u|}\right) < \frac{q}{12\lambda_1^2},$$

since Claim 4.8 leads to

$$|u'_2 - u'| \geq \delta_{D'}(u') \geq \frac{q}{12\lambda_1^2} \delta_{G'}(u').$$

This obvious contradiction completes the proof.

Now, we are ready to get a contradiction to the contrary assumption (4.18). By Claim 4.5 and (4.20), we have

$$\log\left(1 + \frac{\ell(\gamma'[w'_1, w'_2])}{\min\{\delta_{D'}(w'_1), \delta_{D'}(w'_2)\}}\right) \geq \log\left(1 + \frac{\lambda_0 - 24c\lambda_1^2}{24c\lambda_1^2}\right).$$

Also, Lemma D implies

$$\begin{aligned} \log\left(1 + \frac{\ell(\gamma'[w'_1, w'_2])}{\min\{\delta_{D'}(w'_1), \delta_{D'}(w'_2)\}}\right) &\leq \ell_{k_{D'}}(\gamma'[w'_1, w'_2]) \leq k_{D'}(w'_1, w'_2) + \varepsilon \\ &\leq Mk_D(w_1, w_2) + C + 1 \\ &\leq 4b_2^2 M \log\left(1 + \frac{|w_1 - w_2|}{\min\{\delta_D(w_1), \delta_D(w_2)\}}\right) + C + 1. \end{aligned}$$

Furthermore, Claim 4.9 gives

$$\min\{\delta_D(w_1), \delta_D(w_2)\} \geq \frac{q\varphi^{-1}(\log \frac{3}{2})\eta^{-1}(\frac{q}{12\lambda_1^2})}{6(c+1)} \min\{\delta_G(w_1), \delta_G(w_2)\},$$

and Lemma 8(1) leads to

$$k_G(w_1, w_2) \leq Mk_{G'}(w'_1, w'_2) + C.$$

Consequently, it follows from Lemma D that

$$\begin{aligned}
\log \left(1 + \frac{\lambda_0 - 24c\lambda_1^2}{24c\lambda_1^2} \right) &\leq \frac{24b_2^2(c+1)M^2}{q\varphi^{-1}(\log \frac{3}{2})\eta^{-1}(\frac{q}{12\lambda_1^2})} k_{G'}(w'_1, w'_2) \\
&+ \frac{24b_2^2(c+1)CM}{q\varphi^{-1}(\log \frac{3}{2})\eta^{-1}(\frac{q}{12\lambda_1^2})} + C + 1 \\
&\leq \frac{96b_1^2b_2^2(c+1)CM^2}{q\varphi^{-1}(\log \frac{3}{2})\eta^{-1}(\frac{q}{12\lambda_1^2})} \log \left(1 + \frac{|w'_1 - w'_2|}{\min\{\delta_{G'}(w'_1), \delta_{G'}(w'_2)\}} \right) \\
&+ \frac{24b_2^2(c+1)CM}{q\varphi^{-1}(\log \frac{3}{2})\eta^{-1}(\frac{q}{12\lambda_1^2})} + C + 1.
\end{aligned}$$

Since (4.19) and Claim 4.5 lead to

$$\begin{aligned}
|w'_1 - w'_2| &\leq |w'_1 - x'| + |x' - y'| + |y' - w'_2| \leq (24c\lambda_1^2 + 1)|x' - y'| \\
&\leq \left(4\lambda_1^2 + \frac{1}{6c} \right) \min\{\delta_{G'}(w'_1), \delta_{G'}(w'_2)\}.
\end{aligned}$$

Finally, we see that

$$\begin{aligned}
\log \left(1 + \frac{\lambda_0 - 24c\lambda_1^2}{24c\lambda_1^2} \right) &\leq \frac{96b_1^2b_2^2(c+1)CM^2}{q\eta^{-1}(\frac{q\varphi^{-1}(\log \frac{3}{2})}{12\lambda_1^2})} \log \left(1 + 4\lambda_1^2 + \frac{1}{6c} \right) \\
&+ \frac{24b_2^2(c+1)CM}{q\varphi^{-1}(\log \frac{3}{2})\eta^{-1}(\frac{q}{12\lambda_1^2})} + C + 1 \\
&= \rho,
\end{aligned}$$

which is the desired contradiction since $\lambda_0 \geq 48c\lambda_1^2e^\rho$. Hence Theorem 2 holds. \square

5. THE PROOF OF THEOREM 3

We start with two notations. Let $f : X \rightarrow Y$ be a homeomorphism between two metric spaces, and let x be a non-isolated point of X . We write

$$L(x, f) = \limsup_{y \rightarrow x} \frac{|y' - x'|}{|y - x|} \quad \text{and} \quad l(x, f) = \liminf_{y \rightarrow x} \frac{|y' - x'|}{|y - x|}.$$

Suppose G denotes a proper subdomain in X . If $|\cdot| = k_G(\cdot)$, then we denote $L(x, f)$ and $l(x, f)$ by $L_{k_G}(x, f)$ and $l_{k_G}(x, f)$, respectively.

Now, we are going to show three lemmas. The first lemma is about the comparison of the quantities $L(x, f)$ and $L_{k_G}(x, f)$ (resp. $l(x, f)$ and $l_{k_G}(x, f)$).

Lemma 9. *Suppose that X is c -quasiconvex and Y is c' -quasiconvex, and that $G \subsetneq X$ and $G' \subsetneq Y$ are domains. If $f : G \rightarrow G'$ is continuous, then*

$$\frac{L(x, f)\delta_G(x)}{6c\delta_{G'}(x')} \leq L_{k_G}(x, f) \leq 6c' \frac{L(x, f)\delta_G(x)}{\delta_{G'}(x')}$$

and

$$\frac{l(x, f)\delta_G(x)}{6c\delta_{G'}(x')} \leq l_{k_G}(x, f) \leq 6c' \frac{l(x, f)\delta_G(x)}{\delta_{G'}(x')}.$$

Proof. By symmetry, it suffices to prove the first chain of inequalities in the lemma. On the one hand, by Lemma B, we have

$$\begin{aligned} L_{k_G}(x, f) &= \limsup_{y \rightarrow x} \frac{k_{G'}(x', y')}{k_G(x, y)} = \limsup_{y \rightarrow x} \left(\frac{k_{G'}(x', y')}{|x' - y'|} \frac{|x' - y'|}{|x - y|} \frac{|x - y|}{k_G(x, y)} \right) \\ &\leq L(x, f) \frac{6c'\delta_G(x)}{\delta_{G'}(x')}. \end{aligned}$$

On the other hand, again by Lemma B, we know

$$\begin{aligned} L(x, f) &= \limsup_{y \rightarrow x} \frac{|x' - y'|}{|x - y|} \\ &= \limsup_{y \rightarrow x} \left(\frac{|x' - y'|}{k_{G'}(x', y')} \frac{k_{G'}(x', y')}{k_G(x, y)} \frac{k_G(x, y)}{|x - y|} \right) \\ &\leq L_{k_G}(x, f) \frac{6c\delta_{G'}(x')}{\delta_G(x)}. \end{aligned}$$

Hence the proof is complete. \square

The next lemma is a characterization for a homeomorphism from X to Y to be M -QH in terms of $l_{k_G}(x, f)$ and $L_{k_G}(x, f)$.

Lemma 10. *Suppose $f : G \rightarrow G'$ is a homeomorphism, where G denotes a proper subdomain of X . Then f is M -QH if and only if*

$$\frac{1}{M} \leq l_{k_G}(x, f) \leq L_{k_G}(x, f) \leq M.$$

Proof. Since the spaces (G, k_G) and $(G', k_{G'})$ are τ -quasiconvex for all $\tau > 1$ (see [23, Lemma 2.5]), we easily know from [37, Lemma 5.5] that the lemma holds. \square

We are ready to prove the main lemma in this section.

Lemma 11. *Suppose that X is c -quasiconvex and Y is c' -quasiconvex, and that $G \subsetneq X$ and $G' \subsetneq Y$ are domains. If $f : G \rightarrow G'$ is M -QH, then f is fully M' -QH with $M' = 216(cc'M)^2$. That is, f is φ -FQC with $\varphi(t) = M't$.*

Proof. Let D be an arbitrary domain in G . Fix a point x in D . By symmetry and Lemma 10, it suffices to show that

$$(5.1) \quad L_{k_D}(x, f) \leq M'.$$

Since f is M -QH, Lemma 10 gives $L_{k_G}(x, f) \leq M$. Hence, to prove (5.1), by Lemma 9, it suffices to show that

$$(5.2) \quad \frac{\delta_D(x)}{\delta_{D'}(x')} \leq 6cc'M \frac{\delta_G(x)}{\delta_{G'}(x')}.$$

If $\delta_{D'}(x') \geq \frac{1}{6cc'M} \delta_{G'}(x')$, then (5.2) is obvious. So we assume in the following that

$$\delta_{D'}(x') < \frac{1}{6cc'M} \delta_{G'}(x').$$

Let

$$0 < \epsilon < \frac{1}{6cc'M} \delta_{G'}(x') - \delta_{D'}(x'),$$

and let $y' \in \partial D'$ be such that

$$(5.3) \quad |y' - x'| < \delta_{D'}(x') + \epsilon < \frac{1}{6cc'M} \delta_{G'}(x').$$

Then by Lemma 3, we know $y' \in G'$, and thus, Lemma B leads to

$$k_{G'}(x', y') \leq 3c' \frac{|x' - y'|}{\delta_{G'}(x')} \leq \frac{1}{2cM},$$

which shows that

$$k_G(x, y) \leq Mk_{G'}(x', y') \leq \frac{1}{2c}.$$

Hence again by Lemma B, together with (5.3), we have

$$\begin{aligned} \delta_D(x) &\leq |x - y| \leq 2k_G(x, y)\delta_G(x) \leq 2Mk_{G'}(x', y')\delta_G(x) \leq 6c'M \frac{|x' - y'|}{\delta_{G'}(x')} \delta_G(x) \\ &\leq 6c'M \frac{\delta_{D'}(x') + \epsilon}{\delta_{G'}(x')} \delta_G(x). \end{aligned}$$

Letting $\epsilon \rightarrow 0$ gives (5.2). Hence the proof of Lemma 11 is complete. \square

The proof of Theorem 3. Obviously, the proof of Theorem 3 easily follows from Theorem 2 and Lemma 11.

REFERENCES

1. L. V. AHLFORS, Quasiconformal reflections, *Acta Math.*, **109** (1963), 291–301.
2. G. D. ANDERSON, M. K. VAMANAMURTHY AND M. VUORINEN, Dimension-free quasiconformal distortion in n -space. *Trans. Amer. Math. Soc.*, **297** (1986), 687–706.
3. A. F. BEARDON, The Apollonian metric of a domain in \mathbb{R}^n , In: *Quasiconformal mappings and analysis*, Springer-Verlag, 1998, 91–108.
4. A. BEURLING AND L. V. AHLFORS, The boundary correspondence under quasiconformal mappings, *Acta Math.*, **96** (1956), 125–142.
5. M. BONK, J. HEINONEN AND P. KOSKELA, Uniformizing Gromov hyperbolic domains, *Asterisque* **270** (2001), 1–99.
6. ST. M. BUCKLEY AND D. A. HERRON, Uniform spaces and weak slice spaces, *Conform. Geom. Dyn.*, **11** (2007), 191C206 (electronic).
7. ST. M. BUCKLEY, D. A. HERRON AND X. XIE, Metric space inversions, quasihyperbolic distance, and uniform spaces, *Indiana Univ. Math. J.*, **57** (2008), 837–890.
8. ST. M. BUCKLEY, P. KOSKELA AND G. LU, Boman equals John. In: *XVI Rolf Nevanlinna Colloquium: proceedings of the international conference held in Joensuu, Finland, August 1-5, 1995*. W. de Gruyter, Berlin, New York, pp. 91–99.
9. J. L. FERNÁNDEZ, J. HEINONEN AND O. MARTIO, Quasilines and conformal mappings, *J. Analyse Math.*, **52** (1989), 117–132.
10. F. W. GEHRING, Uniform domains and the ubiquitous quasidisks, *Jahresber. Deutsch. Math. Verein.*, **89** (1987), 88–103.
11. F. W. GEHRING AND O. MARTIO, Quasiextremal distance domains and extension of quasiconformal mappings, *J. Analyse Math.*, **45** (1985), 181–206.
12. F. W. GEHRING AND B. G. OSGOOD, Uniform domains and the quasi-hyperbolic metric, *J. Analyse Math.*, **36** (1979), 50–74.

13. F. W. GEHRING AND B. P. PALKA, Quasiconformally homogeneous domains, *J. Analyse Math.*, **30** (1976), 172–199.
14. P. HÄSTÖ, The Apollonian metric: uniformity and quasiconvexity, *Ann. Acad. Sci. Fenn. Ser. Math.*, **28** (2003), 385–414.
15. J. HEINONEN, Quasiconformal mappings onto John domains, *Rev. Mat. Iberoamericana*, **5** (1989), 97–123.
16. J. HEINONEN, *Lectures on analysis on metric spaces*, Springer, 2001.
17. J. HEINONEN AND P. KOSKELA, Definitions of quasiconformality, *Invent. Math.*, **120** (1995), 61–79.
18. J. HEINONEN AND P. KOSKELA, Quasiconformal maps in metric spaces with controlled geometry, *Acta Math.*, **181** (1998), 1–61.
19. D. HERRON, N. SHANMUGALINGAM AND X. XIE, Uniformity from Gromov hyperbolicity, *Illinois J. Math.*, **52** (2008), 1065–1109.
20. M. HUANG AND Y. LI, Decomposition properties of John domains in real normed vector spaces, *J. Math. Anal. Appl.*, **388** (2012), 191–197.
21. M. HUANG, Y. LI, S. PONNUSAMY AND X. WANG, The quasiconformal subinvariance property of John domains in \mathbb{R}^n and its applications, To appear in *Math. Ann.*.
22. M. HUANG, S. PONNUSAMY, A. RASILA AND X. WANG, On quasisymmetry of quasiconformal mappings and its applications, arXiv: 1207.4360v2 [math. CV], 2013.
23. M. HUANG, A. RASILA, X. WANG AND Q. ZHOU, The characterizations of free quasiconformal mappings on metric spaces, arXiv: 1409.5631 [math. CV], 2014.
24. M. HUANG, M. VUORINEN AND X. WANG, On the subinvariance of uniform domains in Banach spaces, *J. Math. Anal. Appl.*, **407** (2013), 527–540.
25. X. HUANG AND J. LIU, Quasihyperbolic metric and quasisymmetric mappings in metric spaces, To appear in *Trans. Amer. Math. Soc.*.
26. F. JOHN, Rotation and strain, *Comm. Pure Appl. Math.*, **14** (1961), 391–413.
27. P. W. JONES, Extension theorems for BMO, *Indiana Univ. Math. J.*, **29** (1980), 41–66.
28. Y. LI AND X. WANG, Unions of John domains and uniform domains in real normed vector spaces, *Ann. Acad. Sci. Fenn. Ser. A I Math.*, **35** (2010), 627–632.
29. O. MARTIO, Definitions of uniform domains, *Ann. Acad. Sci. Fenn. Ser. A I Math.*, **5** (1980), 197–205.
30. O. MARTIO AND J. SARVAS, Injectivity theorems in plane and space, *Ann. Acad. Sci. Fenn. Ser. A I Math.*, **4** (1978), 383–401.
31. A. RASILA AND J. TALPONEN, On quasihyperbolic geodesics in Banach spaces, *Ann. Acad. Sci. Fenn. Math.*, **39** (2014), 163–173.
32. P. TUKIA AND J. VÄISÄLÄ, Quasisymmetric embeddings of metric spaces, *Ann. Acad. Sci. Fenn. Ser. A I Math.*, **5** (1980), 97–114.
33. J. TYSON, Quasiconformality and quasisymmetry in metric measure spaces, *Ann. Acad. Sci. Fenn. Math.*, **23** (1998), 525–548.
34. J. VÄISÄLÄ, Quasimöbius maps, *J. Analyse Math.*, **44** (1985), 218–234.
35. J. VÄISÄLÄ, Uniform domains, *Tohoku Math. J.*, **40** (1988), 101–118.
36. J. VÄISÄLÄ, Free quasiconformality in Banach spaces. I, *Ann. Acad. Sci. Fenn. Ser. A I Math.*, **15** (1990), 355–379.
37. J. VÄISÄLÄ, Free quasiconformality in Banach spaces. II, *Ann. Acad. Sci. Fenn. Ser. A I Math.*, **16** (1991), 255–310.
38. J. VÄISÄLÄ, Free quasiconformality in Banach spaces. III, *Ann. Acad. Sci. Fenn. Ser. A I Math.*, **17** (1992), 393–408.
39. J. VÄISÄLÄ, Free quasiconformality in Banach spaces. IV, *Analysis and Topology*, 697–717, *World Sci. Publ., River Edge, N. J.*, 1998.
40. J. VÄISÄLÄ, Relatively and inner uniform domains, *Conformal Geom. Dyn.*, **2** (1998), 56–88.

- 41. J. VÄISÄLÄ, The free quasiworld: freely quasiconformal and related maps in *Quasiconformal geometry and dynamics (Lublin 1996)*, *Banach Center Publications*, Vol. 48, Polish Academy of Science, Warsaw, 1999, 55-118.
- 42. J. VÄISÄLÄ, Gromov hyperbolic spaces, *Expo. Math.*, **23** (2005), 187-231.
- 43. X. XIE, Quasimöbius maps preserve uniform domains, *Ann. Acad. Sci. Fenn. Ser. A I Math.*, **32** (2007), 481-495.

YAXIANG LI, COLLEGE OF SCIENCE, CENTRAL SOUTH UNIVERSITY OF FORESTRY AND TECHNOLOGY, CHANGSHA, HUNAN 410004, PEOPLE'S REPUBLIC OF CHINA
E-mail address: yaxiangli@163.com

MANZI HUANG, DEPARTMENT OF MATHEMATICS, HUNAN NORMAL UNIVERSITY, CHANGSHA, HUNAN 410081, PEOPLE'S REPUBLIC OF CHINA
E-mail address: mzhuang79@yahoo.com.cn

XIANTAO WANG, DEPARTMENT OF MATHEMATICS, SHANTOU UNIVERSITY, SHANTOU, GUANGDONG 515063, PEOPLE'S REPUBLIC OF CHINA
E-mail address: xtwang@stu.edu.cn

QINGSHAN ZHOU, DEPARTMENT OF MATHEMATICS, HUNAN NORMAL UNIVERSITY, CHANGSHA, HUNAN 410081, PEOPLE'S REPUBLIC OF CHINA
E-mail address: q476308142@qq.com